Single Commodity Flow Algorithms for Lifts of Graphic and Cographic Matroids

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Consider a binary matroid M given by its matrix representation. We show that if M is a lift of a graphic or a cographic matroid, then in polynomial time we can either solve the single commodity flow problem for M or find an obstruction for which the Max-Flow Min-Cut relation does not hold. The key tool is an algorithmic version of Lehman's Theorem for the set covering polyhedron.

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Chapter 1

Introduction

1.1 Multi-Commodity Flow in Graphs

1.1.1 Flow Definitions

A graph $G = (V, E)$, weight function $w \in \mathbb{Z}_+^E$, and set of demand edges $\Sigma \subseteq E$ determine a multi-commodity flow instance. Each $d \in \Sigma$ is a *demand edge* with demand w_d while each $e \in E - \Sigma^1$ is a *capacity edge* with capacity w_e. Given a graph $G = (V, E)$, $C \subseteq E$ is a *circuit* of G if $G[C]$ is connected and 2-regular. Let $\mathcal{C}_1$ be the set of circuits of G intersecting Σ in exactly one demand edge. Call $y \in \mathbb{R}_+^{\mathcal{C}_1}$ a *(fractional) w-flow* if it satisfies

(1) *demand constraints*: for each $d \in \Sigma$: $\qquad \sum(y_C : d \in C \in \mathcal{C}_1) = w_d,$ (1.1)

(2) *capacity constraints*: for each $e \in E \setminus \Sigma$: $\qquad \sum(y_C : e \in C \in \mathcal{C}_1) \leq w_e.$ (1.2)

A *cut* of G is a set of edges $\delta(S) \subseteq E$ for $S \subseteq V$ where $\delta(S)$ is defined as the set of edges with exactly one endpoint in S. That is, $\delta(S) := \{e = xy : x \in S, y \notin S\}$. In the future, $\delta(v)$ will denote the set $\delta(\{v\})$. The *cut condition* is a necessary condition for the existence of a fractional w-flow. It states that the demand across any cut should not exceed the capacity across the cut. Letting $w(F) := \sum(w_e : e \in F)$ for $F \subseteq E$, the cut condition states that for all $S \subseteq V$, $w(\delta(S) - \Sigma) \geq w(\delta(S) \cap \Sigma)$.

A *signed graph* is a pair (G, Σ) where G is a graph and $\Sigma \subseteq E(G)$. We say $B \subseteq E$ is odd if $|B \cap \Sigma|$ is odd. We say $B \subseteq E$ is even if $|B \cap \Sigma|$ is even. In particular, edges in Σ

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[1]Here '$-$' indicates set difference and '$A - a$' denotes $A - \{a\}$

are odd. If C is a circuit of G and $|C \cap \Sigma|$ is odd, we call C an *odd circuit* of (G, Σ). Given (G, Σ), updating Σ to $\Sigma' = \Sigma \triangle \delta(S)$ for $S \subseteq V$ does not change the set of odd circuits in (G, Σ). [2] Any Σ' that can be obtained in this way is called a *signature* of (G, Σ). The cut condition can be reformulated using signatures as follows. For a proof, see Remark 1.2.1.

Remark 1.1.1. *The cut condition holds for (G, Σ) if and only if Σ is a minimum weight signature of (G, Σ).*

If $w \in \mathbb{Z}_+^E$ satisfies the cut condition, then triple (G, Σ, w) defines two multi-commodity flow problems. The *fractional multi-commodity flow problem* asks if there exists a fractional w-flow $y \in \mathbb{R}_+^{\mathcal{C}_1}$. The *integer multi-commodity flow problem* asks if there exists an integer w-flow $y \in \mathbb{Z}_+^{\mathcal{C}_1}$.

Finding an integer w-flow is equivalent to assigning to each $C \in \mathcal{C}$ an integer multiplicity y_C subject to two restrictions: (1) the total multiplicities of circuits through $e \notin \Sigma$ must not exceed w_e, and, (2) the total multiplicity of circuits through $d \in \Sigma$ must equal w_d. If $w_e = 1$ for all $e \in E$, the instance is said to have *unit weights* and a w-flow is called an $\mathbb{1}$-*flow*. Finding a $\mathbb{1}$-flow corresponds to choosing for each $d \in \Sigma$ a circuit whose intersection with Σ is exactly $\{d\}$, with the restriction that the circuits chosen are pairwise disjoint. This viewpoint foreshadows the connection to packing discussed in Section 2.4.2. Figure 1.1 demonstrates some of the definitions introduced thus far.
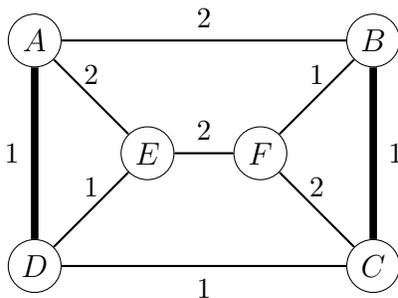
Figure 1.1: Multi-Commodity Flow Instance: Demand edges are the thick edges (AD and BC). Edge e is labelled by weight w_e. If $C_1 = \{AE, DE, AD\}$, $C_2 = \{BC, CF, BF\}$ and $C_3 = \{BC, CF, EF, AE, AB\}$, then $C_1 \in \mathcal{C}_{AD}$ and $C_2, C_3 \in \mathcal{C}_{BC}$. The assignment $y_{C_1} = 1$, $y_{C_2} = y_{C_3} = 1/2$ specifies a fractional w-flow. The assignment $y_{C_1} = 1$, $y_{C_2} = 1$ specifies an integer w-flow. Note that C_1 and C_2 are disjoint circuits of G.

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[2]Here $\triangle$ denotes symmetric difference; $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

The role of (G, Σ) in the existence of multi-commodity flows for all weight functions is of interest. A weight function $w \in \mathbb{Z}_+^E$ is called *Eulerian* if $w(\delta(v))$ is even for all $v \in V$. The role of (G, Σ) in the existence of integer flows for all Eulerian weight functions is also of interest. We look for sufficient conditions; what guarantees regarding the structure of (G, Σ) imply a flow exists for all weight functions $w \in \mathbb{Z}_+^E$? Although this may seem less general, it ensures our results speak only to the structure of (G, Σ). Given (G, Σ) and w satisfying the cut condition, consider the following problems.

1. *Fractional Flow Problem*: Does there exist a fractional w-flow?
2. *Integer Flow Problem*: Does there exist an integer w-flow?
3. *Eulerian Integer Flow Problem*: If w is Eulerian, does there exist an integer w-flow?

Clearly, if there is an integer w-flow for every $w \in \mathbb{Z}_+^E$ satisfying the cut condition, necessarily there is a fractional w-flow for all such w. In fact, if there exists an integer w-flow for every Eulerian $w \in \mathbb{Z}_+^E$ satisfying the cut condition, there is a fractional w-flow for all w satisfying the cut condition. To see this, note that given $w \in \mathbb{Z}_+^E$, $2w$ is necessarily an Eulerian weight function and also satisfies the cut condition. By assumption, there exists a $2w$-flow y; $\frac{1}{2}y$ is a fractional w-flow.

It may not be immediately obviously that there exist "bad instances". A bad instance for the fractional flow problem is a triple (G, Σ, w) where w satisfies the cut condition but (G, Σ) does not admit a fractional w-flow. Likewise, a bad instance for the integer flow problem is a triple (G, Σ, w) where w satisfies the cut condition but (G, Σ) does not admit an integer w-flow. The two important signed graphs introduced next demonstrate that such instances exist.

1.1.2   Bad Instances

Odd-K_n

Given signed graph (G, Σ), the set of signed graphs (G, Σ') such that $\Sigma' = \Sigma \triangle \delta(S)$ for $S \subseteq V$ form an equivalence class. Note that (G, Σ) and (G, Σ') are equivalent if and only if Σ' is a signature of (G, Σ). We may refer to (G, Σ') as a *resigning* of (G, Σ). We let odd-K_n denote the equivalence class containing $(K_n, E(K_n))$ where K_n denotes the complete graph on n vertices.

Odd-K_4

Consider a flow problem on the graph K_4. The two demand edges (Σ_4) are a pair of non-adjacent edges in K_4. The weight function is unit: $w_e = 1$ for all $e \in E$. This instance is illustrated in Figure 1.2. Resigning on $\delta(\{A, B\})$ shows this signed graph is in the equivalence class odd-K_4.
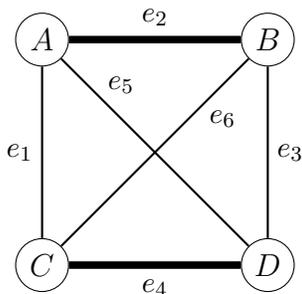
Figure 1.2: Graph is K_4. Each edge is labelled by its name. Thick edges are demand edges.

The circuits intersecting Σ_4 exactly in the edge e_2 are $\mathcal{C}_{e_2} = \{\{e_2, e_3, e_5\}, \{e_1, e_2, e_6\}\}$. The circuits intersecting Σ_4 exactly in the edge e_4 are $\mathcal{C}_{e_4} = \{\{e_1, e_4, e_5\}, \{e_3, e_4, e_6\}\}$. Assigning $1/2$ to each $C \in \mathcal{C}_1$, we obtain a fractional flow for the unit weight function. That is, $y_{\{e_2,e_3,e_5\}} = y_{\{e_1,e_2,e_6\}} = y_{\{e_1,e_4,e_5\}} = y_{\{e_3,e_4,e_5\}} = 1/2$ is a $\mathbb{1}$-flow. Since the cut condition is necessary for the existence of a fractional w-flow, the fractional $\mathbb{1}$-flow y demonstrates that the cut condition holds.

This instance, however, does not have an integer flow. Finding an integer flow is equivalent to finding $C_2 \in \mathcal{C}_{e_2}$ and $C_4 \in \mathcal{C}_{e_4}$ such that $C_2 \cap C_4 = \emptyset$. It is straightforward to verify that no such C_2 and C_4 exist.

Odd-K_5

Consider a flow problem on the graph K_5. There are four demand edges: any triangle plus the unique remaining edge not adjacent to a triangle edge. Again consider the unit weight function. Note that $w(\delta(v)) = |\delta(v)| = 4$ for all $v \in V$; the weight function is Eulerian. This instance is illustrated in Figure 1.3. Resigning on $\delta(\{A, B, C\})$ shows this signed graph is in the equivalence class odd-K_5.
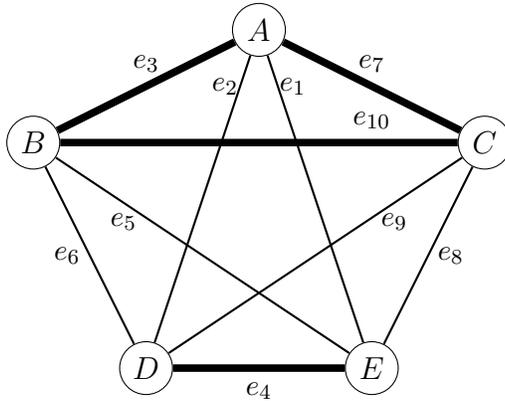
Figure 1.3: Graph is K_5. Each edge is labelled by its name. Thick edges are demand edges.

It is straightforward (but tedious) to verify that the cut condition holds for this instance. However, there does not exist a fractional flow. The total demand is $w(\Sigma_5) = 4$ and the total capacity is $w(E \setminus \Sigma_5) = 6$. Since each circuit using exactly one demand edge has length exactly three, it requires 2ϵ units of capacity to satisfy ϵ units of demand and so 8 units of capacity are required to route the 4 units of demand. Only 6 units of capacity is available and therefore no fractional flow exists. Obviously, if there is no fractional flow then there is no integer flow. In sum, this is an instance with an Eulerian weight function, but no fractional or integer flow.

1.1.3 Minors

We return to the question of interest: what conditions can be imposed on (G, Σ) to guarantee the existence of integer or fractional flows for all (or all Eulerian) weight functions? Having established this is a reasonable question, we first require a mechanism for ascribing structure to (G, Σ).

Graph minors are frequently used to classify the structure of (unsigned) graph G. Given $G = (V, E)$, there are two minor operations: deletion and contraction. *Deleting* edge $e \in E$ is done by removing e; the resulting graph is denoted by $G \backslash e$. In other words, we define $G \backslash e$ as $(V, E - e)$. *Contracting* edge $e \in E$ is done by deleting e and identifying its endpoints; the resulting graph is denoted by G/e. For $e = xy$, we define G/e as $(V \cup \{z\} - \{x, y\}, E')$ with E' defined as follows: if $f \in E$ and neither endpoint of f is x, y, then $f \in E'$; if $f \in E$

and $f = sx$ or $f = xy$ then $sz \in E'$. A *minor* of G is any graph obtained from G by a sequence of contractions and deletions.

Graph minor operations can be thought to act on the circuits of the graph. Deleting edge e results in removing from G all the circuits that used edge e and otherwise preserving the set of circuits. Contracting edge e results in removing the edge e from all the circuits in G using edge e and otherwise preserving the set of circuits. It is straightforward to verify that deletions and contractions can be performed in any order. Namely, if G' is obtained from G by deleting edges in I and contracting edges in J in some order, then we can unambiguously denote G' by $G/I \setminus J$. One way to classify a graph is according to whether it contains a certain fixed minor; G contains G' as a minor if and only if there exists $I, J \subseteq E(G)$ such that $G' = G/I \setminus J$.

Although graph minors characterize the structure of graph G, we are interested in the structure of the signed graph (G, Σ). The idea of minors in graphs can be extended to minors in signed graphs. Recall that given (G, Σ), updating Σ to $\Sigma' = \Sigma \triangle \delta(S)$ for $S \subseteq V$ does not change the set of odd circuits in (G, Σ). We call this operation *resigning*. Given (G, Σ) and $e \in E$, we define $(G, \Sigma) \setminus e$ as $(G \setminus e, \Sigma - e)$ and call this operation *deletion*. Given (G, Σ) and $e \in E$ that is not an odd loop (edge with same endpoints), suppose we resign to obtain Σ' such that $e \notin \Sigma'$. We define $(G, \Sigma)/e$ as $(G/e, \Sigma')$ and call this operation *contraction*. A *minor* of (G, Σ) is any signed graph (G', Σ') obtained from (G, Σ) by a sequence of contractions and deletions. Deleting e in (G, Σ) removes all the circuits using e. Contracting $e \in (G, \Sigma)$ shortens all the circuits using e while maintaining their parity.

Since minor operations can be applied in any order, we denote by $(G, \Sigma)/I \setminus J$ the signed graph obtained by contracting all edges in $I \subseteq E$ and deleting all edges in $J \subseteq E$ (and resigning in whatever way avoids contracting signature edges). Necessarily I does not contain an odd circuit. One way to classify a signed graph is according to whether it contains a certain fixed minor; (G, Σ) contains (G', Σ') as a minor if and only if there exists $I, J \subseteq E(G)$ such that $(G', \Sigma'') = (G, \Sigma)/I \setminus J$ and $\Sigma' = \Sigma'' \triangle \delta_{G'}(U)$ for $U \subseteq V(G')$. Note that this definition does not account explicitly for the resigning operation. If (G', Σ') is a minor of (G, Σ) then so is any resigning of (G', Σ'). In other words, writing "(G', Σ') is a minor of (G, Σ)" means that every resigning of (G', Σ') is a minor of every resigning of (G, Σ).

1.1.4 Existing Results

Having defined minors, we return to sufficient conditions for the existence of flows. One way to give such conditions is by providing a list of minors with the property that whenever

(G, Σ) contains no minor on the list, (G, Σ) has a flow for every $w \in \mathbb{Z}_+^E$. Such lists are known for the flow problems of interest.

Theorem 1.1.2 (Seymour [29]). *Suppose (G, Σ) has no odd-K_4 minor. Then for every $w \in \mathbb{Z}_+^E$ satisfying the cut condition, (G, Σ) has an integer w-flow.*

Theorem 1.1.3 (Guenin [18]). *Suppose (G, Σ) has no odd-K_5 minor. Then for every weight function $w \in \mathbb{Z}_+^E$ satisfying the cut condition, (G, Σ) has a fractional w-flow.*

Theorem 1.1.4 (Geelen, Guenin [13]). *Suppose (G, Σ) has no odd-K_5 minor. Then for every Eulerian weight function $w \in \mathbb{Z}_+^E$ satisfying the cut condition, (G, Σ) has an integer w-flow.*

1.2 Multi-Commodity Flow in Binary Matroids

1.2.1 Matroid Theory Background

A matroid M is a pair $(E, \mathcal{I})$ where $E := E(M)$ is a finite set called the *ground set*, and $\mathcal{I} \subseteq 2^E$ is a set of subsets of E satisfying the following properties:

(I1) $\emptyset \in \mathcal{I}$,

(I2) if $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$, and

(I3) if $X \subseteq E$ and I_1 and I_2 are maximal members of $\{I : I \in \mathcal{I} \text{ and } I \subseteq X\}$, then $|I_1| = |I_2|$.

Set $I \in \mathcal{I}$ is called *independent*; set $I \notin \mathcal{I}$ is called *dependent*. We call a maximally independent subset of E a *basis* of M. Property (I3) shows that all bases have the same size, called the *rank* of the matroid. Let $C \subseteq E$; C is a *circuit* of M if it is an inclusion-wise minimally dependent set. A matroid is uniquely determined by its set of circuits.

A matroid M with ground set $E = \{e_1, \ldots e_n\}$ is said to be representable over field $\mathbb{F}$ if there exists a matrix $A \in \mathbb{F}^{m \times n}$ with columns indexed by $\{e_1 \ldots e_n\}$ such that the independent sets of M coincide exactly with the column sets of A that are linearly independent over $\mathbb{F}$. Matrix A is called a *matrix representation* for M. If M is representable over $GF(2)$, then M is called *binary*. In this paper, we are primarily concerned with binary matroids. In a binary matroid, a *cycle* is any finite symmetric difference of circuits.

The dual of matroid M, denoted M^*, is the matroid with ground set E such that $S \subseteq E$ is independent if and only if there exists a basis B of M such that $S \cap B = \emptyset$. The circuits of M^* are called the *cocircuits* of M. In binary matroids, each cocircuit intersects each

circuit with even parity. A *cocyle* of a binary matroid is any finite symmetric difference of cocircuits.

A matroid M is *graphic* if there exists a graph G such that the circuits of M correspond exactly to the circuits of G. Given graph G, let $M(G)$ denote the affiliated graphic matroid. One possible matrix representation for $M(G)$ is the adjacency matrix of G. A matroid M is *cographic* if there exists a graph G such that circuits of M correspond exactly to the minimal cuts of G. Given graph G, let $M^*(G)$ denote the affiliated cographic matroid. As this notation suggests, $M^*(G)$ is the dual of $M(G)$. All graphic and cographic matroids are binary and thus can be represented by matrices over $GF(2)$. However, these matroids may also be represented by the underlying graph $G = (V, E)$.

1.2.2   Flow Definitions

As matroids are a generalization of graphs, flows in matroids are a generalization of flows in graphs. Throughout this section, the term matroid will refer to a binary matroid. Many of these definitions extend directly to general matroids; Section 2.4.6 justifies the restriction to binary matroids.

A matroid M on ground set E, weight function $w \in \mathbb{Z}_+^E$ and set of demand edges $\Sigma \subseteq E$ determine a flow instance. Let $\mathcal{C}_1$ be the set of circuits in M that intersect Σ in exactly one element. Call $y \in \mathbb{R}_+^{\mathcal{C}_1}$ a *(fractional) w-flow* if it satisfies

(1) *demand constraints*: for each $d \in \Sigma$: $\qquad \sum (y_C : d \in C \in \mathcal{C}_1) = w_d,$

(2) *capacity constraints*: for each $e \in E \setminus \Sigma$: $\qquad \sum (y_C : e \in C \in \mathcal{C}_1) \leq w_e.$

The cut condition is again necessary for the existence of a fractional w-flow. For graphs, the cut condition states that $w(D \cap \Sigma) \leq w(D \setminus \Sigma)$ for all cuts $D = \delta(S)$. Minimal cuts in graphs are the cocircuits of a graphic matroid. The natural extension of the cut condition to arbitrary matroid M states that $w(D \cap \Sigma) \leq w(D \setminus \Sigma)$ for all cocircuits D of M.

A *signed matroid* is a pair (M, Σ) where M is a matroid and $\Sigma \subseteq E(M)$. We say $B \subseteq E$ is odd if $|B \cap \Sigma|$ is odd. We say $B \subseteq E$ is even if $|B \cap \Sigma|$ is even. In particular, edges in Σ are odd. If C is a circuit of M and $|C \cap \Sigma|$ is odd, we call C an *odd circuit* of (M, Σ). Given (M, Σ), updating Σ to $\Sigma' = \Sigma \triangle D$ for cocycle D of M does not change the set of odd circuits. Any Σ' that can be obtained in this way is called a *signature* of (M, Σ). We say (M, Σ) and (M', Σ') are equivalent and call (M', Σ') a *resigning* of (M, Σ) if $M = M'$ and $\Sigma' = \Sigma \triangle D$ for some cocycle D. The cut condition can be reformulated using signatures as follows.

Remark 1.2.1. *The cut condition holds for (M, Σ) if and only if Σ is a minimum weight signature of (M, Σ).*

Proof. The cut condition holds if and only if $w(D \cap \Sigma) \leq w(D \setminus \Sigma)$ for all cocircuits D of M. If the cut condition holds, let $\Sigma' = \Sigma \triangle D$ be an arbitrary signature of (M, Σ). Then

$$w(\Sigma') = w(\Sigma \triangle D) = w(\Sigma \setminus D) + w(D \setminus \Sigma) \geq w(\Sigma \setminus D) + w(D \cap \Sigma) = w(\Sigma)$$

where the inequality ($\geq$) holds as a result of the cut condition. If the cut condition does not hold, then $w(D \cap \Sigma) > w(D \setminus \Sigma)$ for some cocircuit D of M and so the above inequality for $\Sigma' = \Sigma \triangle D$ yields $w(\Sigma') < w(\Sigma)$. It follows that the cut condition holds for (M, Σ) if and only if Σ is a minimum weight signature of (M, Σ). $\qquad\square$

If $w \in \mathbb{Z}_+^E$ satisfies the cut condition, then triple (M, Σ, w) defines two multi-commodity flow problems. The *fractional multi-commodity flow problem* asks if there exists a fractional w-flow $y \in \mathbb{R}_+^{\mathcal{C}_1}$. The *integer multi-commodity flow problem* asks if there exists an integer w-flow $y \in \mathbb{Z}_+^{\mathcal{C}_1}$.

The questions of interest about flows in graphs can be extended to questions about flows in matroids. In particular, we are concerned with signed matroids (M, Σ) for which flows exist for all weight functions $w \in \mathbb{Z}_+^E$. For graphs, the Eulerian condition states that $w(\delta(v))$ is even for all $v \in V$. Since minimal cuts in graphs are the cocircuits of a graphic matroid, the natural extension to matroids defines weight function $w \in \mathbb{Z}_+^{E(M)}$ to be *Eulerian* if $w(D)$ is even for all cocircuits D of M. Given matroid M, $\Sigma \subseteq E$, and $w \in \mathbb{Z}_+^E$ satisfying the cut condition, the following flow problems arise.

1. *Fractional Flow Problem*: Does there exist a fractional w-flow?
2. *Integer Flow Problem*: Does there exist an integer w-flow?
3. *Eulerian Integer Flow Problem*: If w is Eulerian, does there exist an integer w-flow?

Again we are interested in "bad instances"; (M, Σ, w) is a bad instance if w satisfies the cut condition, but there exists no integer flow or no fractional flow.

1.2.3 Bad Instances

Odd-K_4

The matroid $M(K_4)$ is the graphic matroid corresponding to K_4. We will overload notation and refer to the equivalence class containing signed matroid $(M(K_4), E(K_4))$ as *odd-K_4*.

Considering signature $\Sigma_4 = \{e_2, e_4\}$, the odd circuits of $(M(K_4), \Sigma_4)$ are exactly the odd circuits of the complete graph K_4. As per the discussion in Section 1.1.2, the unit weight function satisfies the cut condition, but the signed matroid $(M(K_4), \Sigma_4)$ does not have an integer $\mathbb{1}$-flow.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}, \ \Sigma_4 = \{e_2, e_4\}$$
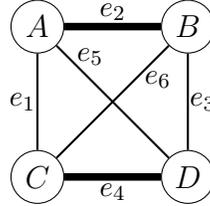
Figure 1.4: Odd-K_4: Matrix and graphic representations for (K_4, Σ_4). Columns of matrix are indexed in order by $\{e_1, e_2, e_3, e_4, e_5, e_6\}$

Odd-K_5

The matroid $M(K_5)$ is the graphic matroid corresponding to K_5. We will refer to the equivalence class containing signed matroid $(M(K_5), E(K_5))$ as *odd-K_5*. Considering signature $\Sigma_5 = \{e_3, e_4, e_7, e_{10}\}$, the odd circuits of $(M(K_5), \Sigma_5)$ are exactly the odd circuits of the complete graph K_5. As per the discussion in Section 1.1.2, the unit weight function is Eulerian and satisfies the cut condition, but the signed matroid $(M(K_5), \Sigma_5)$ does not have a fractional $\mathbb{1}$-flow.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$\Sigma_5 = \{e_3, e_4, e_7, e_{10}\}$
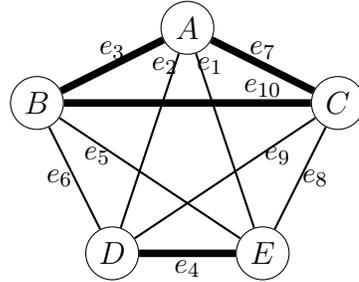
Figure 1.5: Odd-K_5: Matrix and graphic representations for (K_5, Σ_5).

Complements of Cuts of Odd-K_5

Consider the binary matroid T_{10} with matrix representation given by Figure 1.6. Let $\Sigma_{10} = \{e_1, e_7, e_8\}$. The odd circuits of (T_{10}, Σ_{10}) are exactly the complements of the cuts

of K_5 (see Figure 1.5 for the corresponding labelling of K_5). For example, the complement of cut $\delta(\{C\})$ is $C_1 = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and the complement of cut $\delta(\{B, D\})$ is $C_2 = \{e_1, e_6, e_7, e_8\}$. Note that both C_1 and C_2 correspond to minimally dependent column sets of the matrix representation, and both C_1 and C_2 intersect Σ_{10} an odd number of times.

$$
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}, \Sigma_{10} = \{e_1, e_7, e_8\}
$$

Figure 1.6: Matrix and representation for T_{10}.

It can be verified that the unit weight function is Eulerian and satisfies the cut condition. A theorem of Seymour [30] shows that (T_{10}, Σ_{10}) has no fractional flow. To prove this directly, the argument is similar to that for odd-K_5. A proof also arises from a relationship between fractional flow in a matroid and fractional flow in its dual (see Theorem 3.1.1).

Lines of Fano Plane

The binary matroid F_7 is rank 3 matroid with matrix representation given by Figure 1.7. The length 3 circuits of F_7 correspond to the lines of the Fano plane; for example $\{e_1, e_5, e_7\}$ is the length 3 circuit corresponding to the vertical line of the Fano plane. Let signature Σ_7 be any length 3 circuit of F_7; here we arbitrarily choose $\{e_1, e_5, e_7\}$.

$$
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1
\end{bmatrix}, \Sigma_7 = \{e_1, e_5, e_7\}
$$

Figure 1.7: Matrix representation for F_7; length 3 odd circuits are exactly the lines of Fano plane (on right).

Consider the multi-commodity flow instance for signed matroid $(F_7, \{e_1, e_5, e_7\})$ with unit weights. To verify the cut condition, we first note that the cocircuits of F_7 are exactly the complements of the lines of the Fano plane. Since for any cocircuit D, $|D \cap \{e_1, e_5, e_7\}| \leq 2$ and $|D| = 4$, the cut condition holds. Moreover, since all cocycles have an even number of elements, the weights are Eulerian. However, any circuit containing exactly one of $\{e_1, e_5, e_7\}$ has length exactly 3 and so to satisfy ϵ units of demand requires 2ϵ units of capacity. Since $w(\Sigma_7) = 3$, 6 total units of capacity are required, but only $w(E \setminus \Sigma_7) = 4$ units are available. Thus (F_7, Σ_7) has no fractional 1-flow and hence also no integer 1-flow. Resigning on cocircuit $\{e_2, e_3, e_4, e_6\}$ shows $E(F_7)$ is a signature of (F_7, Σ_7).

Postman Sets of Petersen Graph

A *postman set* of graph $G = (V, E)$ is $S \subseteq E$ such that the odd degree vertices of $G[S]$ exactly coincide with the odd degree vertices of G. In Figure 1.8, an example of a postman set of the Petersen graph is given. Let (P_{15}, Σ_{15}) be the signed matroid whose odd circuits are exactly the postman sets of the Petersen graph. Signed matroid (P_{15}, Σ_{15}) has no fractional 1-flow. One way to show this is by deriving a connection between fractional flows and fractional colouring; there is no 3-colouring of the Petersen graph, but there is a fractional 3-colouring. Since colourings correspond to flows, (P_{15}, Σ_{15}) has a fractional 1-flow but no integer 1-flow for the Eulerian unit weight function. For more information about connections between graph colouring and flows, see Section 2.4.4.
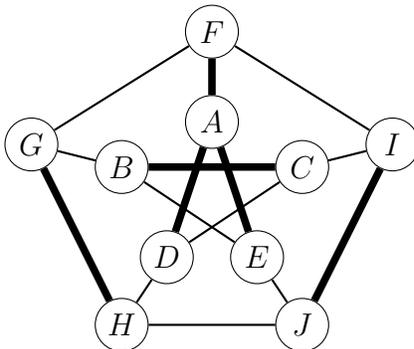
Figure 1.8: Graph is Petersen Graph; thick edges are an example of a postman set.

1.2.4 Minors

The idea of using minors to characterize graphs extends to matroids. Let M be a matroid; the two minor operations that can be applied to M are deletion and contraction. The matroid obtained by *deleting* edge e is denoted by $M \setminus e$ and has circuit set $\{C : C$ is a circuit of M with $e \notin C\}$. The matroid obtained by *contracting* edge e is denoted by M/e and has cycle set $\{C - e : C$ is a cycle of M$\}$. A minor of M is any matroid M' obtained by a sequence of contractions and deletions. Deletions and contractions can be performed in any order; if M' is obtained from M by deleting edges in I and contracting edges in J in some order, then we can unambiguously denote M' by $M/I \setminus J$. One way to classify a matroid is according to whether it contains some fixed minor; M contains M' as a minor if and only if there exists $I, J \subseteq E(M)$ such that $M' = M/I \setminus J$.

Minors in matroids can be extended to minors in signed matroids. There are three minor operations: resigning, deletion and contraction. *Resigning* is done by replacing Σ by $\Sigma' = \Sigma \triangle D$ for D a cocyle. Given (M, Σ) and $e \in E$, we define $(M, \Sigma) \setminus e$ as $(M \setminus e, \Sigma - e)$ and call this operation *deletion*. Given (M, Σ) and $e \in E$ that is not an odd loop (e is a *loop* if $\{e\}$ is dependent), suppose we resign to obtain Σ' such that $e \notin \Sigma'$. We define $(M, \Sigma)/e$ as $(M/e, \Sigma')$ and call this operation *contraction*. A *minor* of (M, Σ) is any signed matroid (M', Σ') obtained from (M, Σ) by a sequence of contractions and deletions.

Deleting edge e removes all the circuits using e and contracting edge e shortens all the circuits using e while maintaining their parity. Since minor operations can be applied in any order, we denote by $(M, \Sigma)/I \setminus J$ the signed matroid obtained by contracting all edges in $I \subseteq E$ and deleting all edges in $J \subseteq E$ (and resigning in whatever way avoids contracting signature edges). Necessarily J does not contain an odd circuit. One way to classify a signed matroid is according to whether it contains a certain fixed minor; (M, Σ) contains (M', Σ') as a minor if and only if there exists $I, J \subseteq E(G)$ such that $(M', \Sigma'') = (M, \Sigma)/I \setminus J$ and $\Sigma' = \Sigma'' \triangle D$ for some cocyle D of (M', Σ''). Note that this definition does not account explicitly for the resigning operation. If (M', Σ') is a minor of (M, Σ) then so is any resigning of (M', Σ'). In other words, writing "(M', Σ') is a minor of (M, Σ)" means that every resigning of (M', Σ') is a minor of every resigning of (M, Σ).

1.2.5 Existing Results and Conjectures

Having defined matroid minors, we return to sufficient conditions for the existence of flows. Theorem 1.1.2 states graphs with no odd-K_4 minor always have integer flows. This theorem extends to integer flows in binary matroids.

Theorem 1.2.2 (Seymour)**.** *Suppose* (M, Σ) *has no odd-K_4 minor. Then for every $w \in \mathbb{Z}_+^E$ satisfying the cut condition, (M, Σ) has an integer w-flow.*

Generalization of Theorems 1.1.3 and 1.1.4 regarding fractional flows and Eulerian integer flows are not known. The Flowing Conjecture [30] posits that the bad instances previously discussed are precisely the obstructions to having a fractional flow for every weight function satisfying the cut condition. The Cycling Conjecture [30] posits that the bad instances previously discussed are precisely the obstructions to having an integer flow for every Eulerian weight function satisfying the cut condition.

Conjecture 1.2.3 (Flowing Conjecture)**.** *If signed matroid (M, Σ) has no odd-K_5, (T_{10}, Σ_{10}), or $(F_7, E(F_7))$ minor, then M has a fractional w-flow for every $w \in \mathbb{Z}_+^E$ satisfying the cut condition.*

Conjecture 1.2.4 (Cycling Conjecture)**.** *If signed matroid (M, Σ) has no odd-K_5, (T_{10}, Σ_{10}), $(F_7, E(F_7))$, or (P_{15}, Σ_{15}) minor, then M has an integer w-flow for every Eulerian $w \in \mathbb{Z}_+^E$ satisfying the cut condition.*

Although the Cycling and Flowing Conjectures are open for general matroids, they are solved for some specific classes of matroids. For example, the Flowing Conjecture has been solved for lifts of graphic and cographic matroids whenever $|\Sigma| = 1$.

1.2.6   Lifts and Projections of Graphic and Cographic Matroids

Consider binary matroid M represented by $0, 1$-matrix A. Let A' be obtained from A by adding some $0, 1$-row outside the row space of A. The binary matroid M' with representation A' is called a *lift* of M. Let A'' be obtained from A by removing some row. Binary matroid M'' with representation A'' is called a *projection* of M.

Lifts of graphic matroids are known as even cycle matroids and lifts of cographic matroids are known as even cut matroids [25]. The Flowing Conjecture (Conjecture 1.2.3)is known to hold for lifts of graphic and cographic matroids whenever Σ consists of exactly one edge [19].

Theorem 1.2.5. *Let M be a lift of a graphic matroid and $e \in E(M)$. If $(M, \{e\})$ has no odd-K_5 or $(F_7, E(F_7))$ minor, then M has a fractional w-flow for every $w \in \mathbb{Z}_+^E$ satisfying the cut condition.*

Theorem 1.2.6. *Let M be a lift of a cographic matroid and $e \in E(M)$. If $(M, \{e\})$ has no (T_{10}, Σ_{10}) or $(F_7, E(F_7))$ minor, then M has a fractional w-flow for every $w \in \mathbb{Z}_+^E$ satisfying the cut condition.*

It can be shown that the dual of a lift of M is a projection of M^*. The Flowing Conjecture (Conjecture 1.2.3) also holds for projections of graphic and cographic matroids. For more details see Section 2.3.

1.3   Results in This Thesis

1.3.1   Algorithmic Point of View

Up to this point, we have been concerned with existential results. Given signed matroid (M, Σ), and w satisfying the cut condition,

1. Does (M, Σ) have a fractional w-flow?
2. Does (M, Σ) have an integer w-flow?
3. If w is Eulerian, does (M, Σ) have an integer w-flow?

Existential questions can be extended to questions about finding w-flows: given (M, Σ) and $w \in \mathbb{Z}_+^{E(M)}$, the aim is to either

1. find a w-flow, or
2. show the cut condition is violated, or
3. find an obstruction.

Here, by obstruction we mean one of the signed graphs appearing in Theorem 1.2.2 and Conjectures 1.2.3 and 1.2.4. Our interest is in finding flows, obstructions, or cut condition violations "efficiently".

1.3.2   Measuring Efficiency

Given some problem with input X, we let $\langle X \rangle$ denote the *encoding length* of the input. For example, the encoding length of a rational number $\alpha = \frac{p}{q}$, is the total number of bits needed to represent both p and q. The encoding length $\langle w \rangle$ of rational vector w is the total encoding length of all entries of w. The encoding length $\langle A \rangle$ of matrix A is the total encoding length of all entries of A. We consider an algorithm for problem with input X *efficient* if it runs in time polynomial in $\langle X \rangle$.

The input for a flow problem is (M, Σ) and $w \in \mathbb{Z}_+^{E(M)}$. The size of the input depends on its representation; for example, matroid M can be provided by its matrix representation A or by a list of its circuits. Since the number of circuits may be exponential in the size of A, an algorithm running in polynomial time in the size of the circuit representation will not necessarily run in polynomial time in $\langle A \rangle$. Accordingly, we need to adopt some convention for measuring the size of a matroid. Throughout this paper, suppose A is the smallest matrix representation of matroid M. Then $\langle A \rangle$ is the size of M. If M is binary, then the entries of A are 0 or 1 and so if A is $m \times n$, then $\langle A \rangle = mn$.

For more details regarding efficiency measures and algorithm analysis see Grötschel et. al. ([16] Chapter 1.2). It is well-known the whenever the cut condition holds, the fractional multi-commodity flow problem can be solved in polynomial time using linear programming. The integer multi-commodity flow problem is polynomial time solvable for $|\Sigma| = 1$ using network flow techniques. However, this problem is known to be NP-hard for $|\Sigma| \geq 2$ [10].

1.3.3   Algorithms Arising from Existential Results

Although we cannot hope for a polynomial time algorithm for solving arbitrary flow instances, restricting the instances in consideration may allow for efficient algorithms. One natural class of problems to consider arises from existential results; does knowing that (M, Σ) has a integer (or fractional) flow for every $w \in \mathbb{Z}_+^E$ satisfying the cut condition help in efficiently finding flows?

By Seymour's Theorem, the instances with integer flows are exactly the instances for which (M, Σ) has no odd-K_4 minor. Using a decomposition theorem, Truemper [33] shows this result leads to an algorithm.

Theorem 1.3.1. *Let (M, Σ) be a signed binary matroid with matrix representation A and let $w \in \mathbb{Z}_+^{E(M)}$ be some weight function. Then in time polynomial in $\langle A \rangle$ and $\langle w \rangle$, one can either*

1. *show the cut condition is violated, or*
2. *find $I, J \subseteq E(M)$ such that $(M, \Sigma)/I \setminus J$ is odd-K_4, or*
3. *find an integer w-flow.*

Note that the above result is (in some sense) stronger than simply being able to find an integer w-flow whenever the cut condition is satisfied and (M, Σ) has no odd-K_4 minor. Given an arbitrary matroid, this results shows that we can either find an integer w-flow or provide a certificate as to why we could not.

Since the Flowing Conjecture (Conjecture 1.2.3) is open for general clutters, we cannot hope to prove a fractional analogue of Theorem 1.3.1. However, in Chapter 4 of this thesis, we show that the analogous result holds whenever M is the lift of a graphic or cographic matroid and $|\Sigma| = 1$.

Theorem 1.3.2. *Let M with representation A be the lift of a graphic matroid and $e \in E(M)$. Consider the flow instance on signed matroid $(M, \{e\})$ with weight function $w \in \mathbb{Z}_+^E$. Then in time polynomial in $\langle A \rangle$ and $\langle w \rangle$ one can either*

1. *show the cut condition is violated, or*
2. *find $I, J \subseteq E(M) - e$ such that $(M, \{e\})/I \setminus J$ is odd-K_5 or $(F_7, E(F_7))$, or*
3. *find a fractional w-flow.*

Theorem 1.3.3. *Let M with representation A be the lift of a cographic matroid and $e \in E(M)$. Consider the flow instance on signed matroid $(M, \{e\})$ with weight function $w \in \mathbb{Z}_+^E$. Then in time polynomial in $\langle A \rangle$ and $\langle w \rangle$ one can either*

1. *show the cut condition is violated, or*
2. *find $I, J \subseteq E(M) - e$ such that $(M, \{e\})/I \setminus J$ is (T_{10}, Σ_{10}) or $(F_7, E(F_7))$, or*
3. *find a fractional w-flow.*

It follows from the extension of the graph minors project to binary matroids that the problem of checking for the presence of the aforementioned obstructions can be done in polynomial time. Hence an existing algorithm for this problem would be to first check for any obstruction and then, if none exists, simply solve the covering linear program to obtain a necessarily integral solution and, hence, a cover. However, the algorithms arising from the matroid minor project are both complicated and impractical because they arise from Ramsey-type arguments and hence have astronomical constants.

The algorithms presented herein use as a tool a theorem of Lehman characterizing minimally non-ideal clutters. Although Lehman's Theorem was non-constructive and applied to minimally non-ideal clutters, a constructive variant can be shown. This is the topic of Chapter 3. As Lehman's Theorem was the basis for many excluded minor results, this constructive version of Lehman's Theorem and a constructive spin on the related excluded minor results leads to the proofs of Theorems 1.3.2 and 1.3.3 appearing in Chapter 4.

An interesting corollary of Theorem 1.3.2 is that the NP-hard maximum cut problem can be solved in polynomial time for graphs that do not contain K_5 as an odd minor. We say K_n is an *odd minor* of graph G if it can be obtained by first deleting edges and then contracting edges on a cut. If K_n is an odd minor of G, then it is a minor, but the converse is not true in general.

Corollary 1.3.4. *Let G be a graph and let $w \in \mathbb{Z}_+^{E(G)}$. Then in time polynomial in $|V(G)| + \langle w \rangle$ we can either,*

1. *find a maximum weight cut of G, or*
2. *find K_5 as an odd minor of G.*

1.4   Overview of Remainder of Thesis

Chapter 2 is a more in-depth coverage of flowing and cycling results; it introduces the related minimax equation and overviews the classes of matroids for which single commodity results are known. The chapter concludes by introducing clutters and showing the relationship to flows in matroids. Chapter 3 starts by introducing the necessary polyhedral background. The remainder of the chapter is dedicated to introducing and proving a constructive version of Lehman's Theorem. Chapter 4 shows how the constructive version of Lehman's Theorem can be applied to lifts of graphic and cographic matroids to prove Theorems 1.3.2 and 1.3.3. Chapter 5 concludes this thesis and addresses future areas of work, including solving Eulerian integer multi-commodity flow in a naturally arising class of signed graphs.

Chapter 2

Flow Results and Relationship to Clutters

2.1 Flows as a Minimax Relation

2.1.1 Definitions and Connections

Given multi-commodity flow instance (M, Σ, w), let $\mathcal{C}$ denote the set of odd circuits of (M, Σ). Consider the following primal-dual pair.

$$
\begin{aligned}
\min \quad & \sum (w_e x_e : e \in E) \\
\text{subject to} \quad & \sum (x_e : e \in C) \geq 1 && \text{for all } C \in \mathcal{C} && (2.1) \\
& x \geq \mathbb{O}
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & \sum (y_C : C \in \mathcal{C}) \\
\text{subject to} \quad & \sum (y_C : e \in C) \leq w_e && \text{for all } e \in E \setminus \Sigma && (2.2) \\
& y \geq \mathbb{O}.
\end{aligned}
$$

The program (2.1) models the problem of finding the minimum weight *fractional cover*. Any integral solution to (2.1) is a *cover* of (M, Σ). The minimal covers of (M, Σ) are exactly

the signatures of (M, Σ). We let $\tau_w^*(M, \Sigma)$ denote the fractional optimal solution to (2.1) and $\tau_w(M, \Sigma)$ denote the integer optimal solution. The program (2.2) models the problem of packing odd circuits. A solution to program (2.2) is called a *fractional w-packing* of odd circuits of (M, Σ). Any integral solution to (2.2) is called an *integer w-packing* of odd circuits of (M, Σ). We let $\nu_w^*(M, \Sigma)$ denote the fractional optimal solution to (2.2) and $\nu_w(M, \Sigma)$ denote the integer optimal solution. The following relationship follows directly from LP duality.

Remark 2.1.1. $\tau_w(M, \Sigma) \geq \tau_w^*(M, \Sigma) = \nu_w^*(M, \Sigma) \geq \nu_w(M, \Sigma)$.

The existence of flows in signed matroids corresponds to packings and covers of equal value. The following two remarks make this statement precise.

Proposition 2.1.2. *If $w \in \mathbb{Z}_+^E$ satisfies the cut condition, then (M, Σ) has a fractional w-flow if and only if $\tau_w(M, \Sigma) = \nu_w^*(M, \Sigma)$.*

Proof. Since w satisfies the cut condition, Σ is a minimum weight signature by Remark 1.2.1. If (M, Σ) has fractional w-flow y, then y is a fractional packing of odd circuits of (M, Σ) of value $w(\Sigma)$. That is $\tau_w(M, \Sigma) = \nu_w^*(M, \Sigma)$.

For the other direction, suppose $\tau_w(M, \Sigma) = \nu_w^*(M, \Sigma)$. This means (2.1) has integral optimal solution of value $w(\Sigma)$ and (2.2) has fractional optimal solution of value $w(\Sigma)$. By complementary slackness,

1. for each $C \in \mathcal{C}$, if $y_C > 0$, then $|C \cap \Sigma| = 1$, and,
2. for each $d \in \Sigma$, $\sum(y_C : e \in C \in \mathcal{C}_d) = w_d$.

Let $\mathcal{C}_1$ again denote the set of odd circuits of (M, Σ) intersecting Σ in exactly one edge. The restriction of y to $\mathcal{C}_1$ is a fractional w-flow for (M, Σ). $\qquad\square$

Proposition 2.1.3. *If $w \in \mathbb{Z}_+^E$ satisfies the cut condition, then (M, Σ) has a integer w-flow if and only if $\tau_w(M, \Sigma) = \nu_w(M, \Sigma)$.*

Proof. Analogous to proof of Proposition 2.1.2. $\qquad\square$

Note that if $\tau_w(M, \Sigma) = \nu_w(M, \Sigma)$ for all $w \in \mathbb{Z}_+^E$, then $\tau_w(M', \Sigma') = \nu_w(M', \Sigma')$ for every minor (M', Σ') of (M, Σ). This follows since deletion corresponds to setting the weight of an edge to 0 and contraction corresponds to setting the weight of an edge to some large number (effectively ∞). Seymour's Theorem (Theorem 1.2.2) regarding integer flows and the Flowing and Cycle Conjectures (Conjectures 1.2.3 and 1.2.4) can be restated as follows.

Theorem 2.1.4 (Seymour). *The equation $\tau_w(M, \Sigma) = \nu_w(M, \Sigma)$ holds for all $w \in \mathbb{Z}_+^E$ if and only if (M, Σ) has no odd-K_4 minor.*

Conjecture 2.1.5 (Flowing Conjecture). *The equation $\tau_w(M, \Sigma) = \nu_w^*(M, \Sigma)$ holds for all $w \in \mathbb{Z}_+^E$ if and only if (M, Σ) has no odd-K_5, (T_{10}, Σ_{10}) or $(F_7, E(F_7))$ minor.*

Conjecture 2.1.6 (Cycling Conjecture). *The equation $\tau_w(M, \Sigma) = \nu_w(M, \Sigma)$ holds for all Eulerian $w \in \mathbb{Z}_+^E$ if and only if (M, Σ) has no odd-K_5, (T_{10}, Σ_{10}), $(F_7, E(F_7))$ or (P_{15}, Σ_{15}) minor.*

For unit weights, finding an integer packing is equivalent to finding pairwise disjoint odd circuits of (M, Σ). This leads to combinatorial interpretations of the excluded minor results. The next two sections investigate Seymour's Theorem (Theorem 2.1.4) and the Cycling Conjecture (Conjecture 2.1.6) for graphic and cographic matroids.

2.1.2 Application to Graphic Matroids

Suppose M is the graphic matroid corresponding to graph G. As indicated in Section 1.1.4, the Cycling Conjecture (Conjecture 2.1.6) holds for graphic matroids. The odd circuits of (M, Σ) are the circuits of G intersecting Σ an odd number of times. Considering unit weights, Seymour's Theorem (Theorem 2.1.4) and the Cycling Conjecture (Conjecture 2.1.6) imply the following minimax relationships.

Corollary 2.1.7. *If (M, Σ) has no odd-K_4 minor, then the minimum size of a signature of (G, Σ) is equal to the maximum number of pairwise disjoint odd circuits in (G, Σ).*

Corollary 2.1.8. *If (M, Σ) has no odd-K_5 minor and $|\delta(\{v\})|$ is even for all $v \in V(G)$, then the minimum size of a signature of (G, Σ) is equal to the maximum number of pairwise disjoint odd circuits in (G, Σ).*

2.1.3 Application to Cographic Matroids

Suppose M is the cographic matroid corresponding to graph G. A result of Seymour [29] shows the Cycling Conjecture (Conjecture 2.1.6) holds for cographic matroids. Given signature Σ, let T denote the odd degree vertices of $G[\Sigma]$. Then the odd circuits of (M, Σ) are the T-cuts of (G, T). A *T-cut* is $\delta(S)$ for $S \subseteq V$ such that $|S \cap T|$ is odd. The covers of (M, Σ) are the T-joins of (G, T). A *T-join* is $J \subseteq E$ such that the odd vertices $G[J]$ are T.

Odd-K_4 has one realization for M cographic (see Figure 2.1). None of the obstructions appearing in the Cycling Conjecture (Conjecture 2.1.6) have a realization in which M is cographic. If the unit weight function is Eulerian, then every cycle of G has even length. Combining these observations, Seymour's Theorem (Theorem 2.1.4) and the Cycling Conjecture (Conjecture 2.1.6) reduce to the following minimax relationships.

Theorem 2.1.9. *Suppose M is the cographic matroid corresponding to G and $\Sigma \subseteq E(G)$ is such that T denotes the odd vertices of $G[\Sigma]$. If (M, Σ) has no odd-K_4 minor, then the size of a minimum of a T-join of (G, T) is equal to the maximum number of pairwise disjoint T-cuts of (G, T).*

Theorem 2.1.10 (Seymour)**.** *If (G, Σ) is bipartite and $\Sigma \subseteq E(G)$ is such that T denotes the odd vertices of $G[\Sigma]$, then the size of a minimum T-join of (G, T) is equal to the maximum number of pairwise disjoint T-cuts of (G, T).*
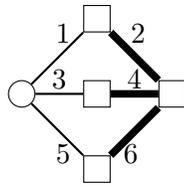
Figure 2.1: Cographic Odd-K_4: Square vertices are T vertices.

2.2 Multi-Commodity to Single Commodity Flow

2.2.1 Equivalence

If $\Sigma = \{e\}$, then (M, Σ) and $w \in \mathbb{Z}_+^E$ specify a *single commodity flow instance*. There is "no difference" between single and multi-commodity flow for the class of all binary matroids.

Let (M, Σ) be a signed matroid and assume $e \notin E(M)$. Let $\mathcal{C}_{odd}$ denote the set of odd circuits of (M, Σ) and $\mathcal{C}_{even}$ denote the set of even circuits of (M, Σ). Let matroid M' have circuit set $\{C : C \in \mathcal{C}_{even}\} \cup \{C \cup \{e\} : C \in \mathcal{C}_{odd}\}$. We call M' the *port representation* of (M, Σ) with port e. By construction, the circuits of M' containing edge e are exactly the odd circuits of M. The first remark below shows that M' is indeed a binary matroid. The second remark shows that multi-commodity flow in (M, Σ) corresponds to single commodity flow in $(M', \{e\})$.

Remark 2.2.1. *Let M' be a port representation of signed binary matroid (M, Σ). Then M' is a binary matroid.*

Proof. Let B be a matrix representation of M^*; the rows of B are characteristic vectors of cycles of M. Obtain B' by appending a column to B that has an 1 in position i if and only if the cycle corresponding to row i of B as an odd cycle of (M, Σ). It can be readily checked that the matroid whose cycle space is spanned by the rows of B' is exactly M'. □

Remark 2.2.2. *Let M' be a port representation of signed binary matroid (M, Σ) with port e. Let $w' \in \mathbb{Z}_+^{E(M)}$ be the weight function defined by $w'_f = w_f$ for all $f \in E(M)$ and $w'_e = w(\Sigma)$. Then*

1. *(M, Σ, w) satisfies the cut condition if and only if $(M', \{e\}, w')$ satisfies the cut condition.*
2. *(M, Σ) has a fractional w-flow if and only if $(M', \{e\})$ has a fractional w'-flow.*
3. *(M, Σ) has an integer w-flow if and only if $(M', \{e\})$ has an integer w'-flow.*

It follows we can restrict ourselves to single commodity flow when considering the class of all binary matroids. However, we cannot use this restriction for fixed classes of binary matroids. For example, suppose we are interested in multi-commodity flow instances in graphic matroids. There is no integer flow whenever (M, Σ) has an odd-K_4 minor. However, the Ford Fulkerson Max Flow Min Cut Theorem [11] shows that any graphic single commodity flow instance has an integer flow whenever the cut condition is satisfied. Switching from multi-commodity flow in graphic M to single commodity flow in M', M' will not necessarily be graphic. In fact, for $(M, \Sigma) = $ odd-K_4, M' cannot be graphic.

To illustrate Remark 2.2.2, consider the multi-commodity flow instance (F_7, Σ_7, w). It is straightforward to verify that $AG(3, 2)$ (with matrix representation given in Figure 2.2) is the port representation of (F_7, Σ_7) with port e_8. In particular, the circuits of $AG(3, 2)$ using edge e_8 are exactly the odd circuits of (F_7, Σ_7) (which are exactly the odd circuits of $(F_7, E(F_7))$). Finding a w-flow for (F_7, Σ_7) is equivalent to finding a w'-flow for $(AG(3, 2), \{e_8\})$ where $w'_f = w'_f$ for all $f \in E - e_8$ and $w'_{e_8} = w(\Sigma_7)$.

It can be shown that T_{11} is a port representation of (T_{10}, Σ_{10}) with port e_{11} and that T_{11}^* is a port representation of odd-K_5 with port e_{11}. For a matrix representation of T_{11}, see Figure 2.3.

23

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0
\end{bmatrix}
$$
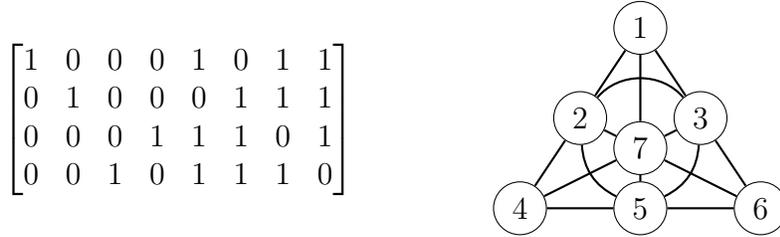
Figure 2.2: Matrix representation for $AG(3,2)$ (left); circuits using e_8 (element represented by the 8th column) correspond to lines of Fano plane (right).

$$
T_{11} : \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}
$$

Figure 2.3: Matrix representation for T_{11}

2.2.2   Reformulating Single Commodity Flow Using e-Paths

Given matroid M and element $e \in E(M)$, an *e-path* is a set of the form $C - e$ where C is a circuit of M containing e. Let $\mathcal{F}$ denote the set of e-paths of M. Given $w \in \mathbb{Z}_+^{E-e}$, we call $\mathcal{P} \subseteq \mathcal{F}$ a *w-packing* of e-paths if $|\{P \in \mathcal{P} : f \in P\}| \leq w_f$ for all $f \in E$.

An *e-cut* is a minimal subset of ground set E that intersects every e-path; that is, $U \subseteq E$ is an e-cut if $|U \cap P| \geq 1$ for all e-paths P and if $U' \subseteq U$ is such that $|U' \cap P| \geq 1$ for all e-paths P then $U' = U$. Whenever M is binary, the e-cuts of M are exactly the e-paths of M^*. A proof of this fact will appear later (see Fact 2.5.2).

Given M, e and w, consider the following primal-dual pair of linear programs.

$$
\begin{aligned}
\min \quad & \sum(w_f x_f : f \in E - e) \\
\text{subject to} \quad & \sum(x_f : f \in P) \geq 1 && \text{for all } e\text{-paths } P && (2.3) \\
& x \geq \mathbb{O}
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & \sum(y_P : P \text{ is an } e\text{-path}) \\
\text{subject to} \quad & \sum(y_P : f \in e\text{-path } P) \leq w_f && \text{for all } f \in E - e && (2.4) \\
& y \geq \mathbb{O}.
\end{aligned}
$$

Note that (2.4) corresponds exactly to (2.2) whenever $\Sigma = \{e\}$ and w_e is maximal such that the cut condition holds. To obtain the correspondence, simply map e-path P to its corresponding odd circuit $P \cup \{e\}$. Since w_e is maximal, there exists an optimal cover for 2.3 that is distinct from $\{e\}$. Likewise (2.3) corresponds exactly to (2.1) whenever $\Sigma = \{e\}$. It follows from Proposition 2.1.2 that we can solve single commodity flow instance $(M, \{e\}, w)$ by solving (2.4) to find a maximum packing of e-paths of M. In Section 2.5, this relationship will be useful to connect statements about flows to statements about clutters.

Recall that we are interested in signed matroids for which the minimax equations ($\tau_w = \nu_w$, $\tau_w = \nu_w^*$) hold at equality for all weight functions. We say $(M, \{e\})$ has the

1. *fractional MFMC property* if for every $w \in \mathbb{Z}_+^E$ there is an integer solution to (2.3) and a solution to (2.4) with the same objective value,
2. *integer MFMC property* if for every $w \in \mathbb{Z}_+^E$ there is an integer solution to (2.3) and an integer solution to (2.4) with the same objective value,
3. *Eulerian integer MFMC property* if for every Eulerian $w \in \mathbb{Z}_+^E$ there is an integer solution to (2.3) and an integer solution to (2.4) with the same objective value.

Seymour's Theorem (Theorem 1.2.2) and the Flowing and Cycling Conjectures (Conjectures 1.2.4 and 1.2.3) can be restated using this terminology. Recall that (P_{15}, Σ_{15}) is the signed binary matroid with odd circuits corresponding to the postman sets of the Petersen graph. Let P_{16} be the port representation of (P_{15}, Σ_{15}) with port e_{16}.

Theorem 2.2.3 (Seymour). *$(M, \{e\})$ has the integer MFMC property if and only if it does not have a minor isomorphic to odd-K_4.*

Conjecture 2.2.4 (Flowing Conjecture). *$(M, \{e\})$ has the fractional MFMC property if and only if it does not have a minor isomorphic to $(AG(3,2), \{e_8\})$, $(T_{11}, \{e_{11}\})$, or $(T_{11}^*, \{e_{11}\})$.*

Conjecture 2.2.5 (Cycling Conjecture). $(M, \{e\})$ *has the Eulerian integer MFMC property if and only if it does not have a minor isomorphic to* $(AG(3, 2), \{e_8\})$, $(T_{11}, \{e_{11}\})$, $(T_{11}^*, \{e_{11}\})$, *or* $(P_{16}, \{e_{16}\})$.

The Flowing and Cycling Conjectures are solved for some classes of single commodity flow instances. For example, they are known to hold whenever M is graphic or cographic.

2.2.3 Single Commodity Flow for Graphic Matroids

Consider a graph G with distinct vertices s, t and edge $e = (s, t)$. Let H be obtained from G by deleting e. Suppose M is the graphic matroid corresponding to G. The e-paths and e-cuts of M correspond respectively to the st-paths and st-cuts of H. An integer solution to (2.6) gives an integer st-flow, and an integer solution to (2.5) gives the characteristic vector of an st-cut. By the Ford Fulkerson Max Flow Min Cut Theorem [11], the value of a maximum st-flow is equal to the size of a minimum st-cut. In other words, the integer MFMC property holds for $(M, \{e\})$ whenever M is a graphic matroid.

2.2.4 Single Commodity Flow for Cographic Matroids

Consider a graph G with distinct vertices s, t and edge $e = (s, t)$. Let H be obtained from G by deleting e. Suppose M is the cographic matroid corresponding to G. The e-paths and e-cuts of M correspond to the st-cuts and st-paths of H respectively. An integer solution to (2.6) gives an integer packing of st-cuts, and an integer solution to (2.5) gives the characteristic vector of an st-path. The Max Work Min Potential Theorem of Duffin [9] states that the size of a maximum packing of st-cuts is equal to the minimum length of an st-path. In other words, the integer MFMC property holds for $(M, \{e\})$ whenever M is a cographic matroid.

2.3 Lifts and Projections of Graphic and Cographic Matroids

2.3.1 Cycle and Cocycle Spaces for Binary Matroids

Let M be a binary matroid. The circuits of M are exactly the sets of minimally linearly dependent columns for some matrix $A \in GF(2)^{m \times n}$. The cycles of M are obtained by

taking the symmetric difference of some finite number of circuits; any cycle C can be partitioned into circuits $C_1, \ldots C_k$ such that $C = C_1 \triangle \ldots \triangle C_k$. The set of cycles of M is called the *cycle space* of the matroid. If C_1 and C_2 are cycles of M, then $C_1 \triangle C_2$ is also a cycle of M. A set of cycles $\mathcal{C}'$ *spans* the cycle space of M if every cycle of M can be expressed as the symmetric difference of cycles in $\mathcal{C}'$. A *basis* for the cycle space of M is any minimal set of cycles spanning the cycle space of M.

Consider matroid M with matrix representation A. The cycle space of M^* is called the *cocycle space* of M. It is spanned by the characteristic sets of the rows of A. (In the future, we will abuse notation and simply say the cycle space of M is spanned by the row space of A; the meaning is clear.) Since $(M^*)^* = M$, if B is a matrix representation for M^*, the row space of B spans the cycle space of M. From Proposition 2.2.23 of [24], the row space of A is orthogonal to the row space of B whenever A is a binary matroid. It follows that if x is the characteristic vector of a cycle of M and y is the characteristic vector of a cocycle of M, $x^T y = 0$.

Consider binary matroid M represented by $0, 1$-matrix A. Let A' be obtained from A by adding some $0, 1$-row outside the row space of A. Recall that the binary matroid M' with representation A' is called a *lift* of M. Let A'' be obtained from A by removing some row of A. Recall that the binary matroid M' with representation A' is called a *projection* of M.

Augmenting A by row x is the same as a projection of the cycle space of M onto the set $\{y : x^T y = 0\}$. By Rank-Nullity, this corresponds to increasing the dimension of the cocycle space of M by 1. That is, if M' denotes the matroid with representation A', then the cycle space of $(M')^*$ has dimension one more than the dimension of the cycle space of M^*. In other words, the dual of a lift of M is a projection of M^*.

2.3.2   Even Cycle Matroids

Even cycle matroids were introduced by Zaslavsky [35] and have a nice interpretation with respect to signed graphs. Given graph G, let A be a matrix representation of $M(G)$. Let A' be obtained by augmenting A with row x^T outside the row space of A where x^T is the characteristic vector of $\Sigma \subseteq E$. The matroid with matrix representation A' is denoted by ecycle(G, Σ) and called the *even cycle matroid* of (G, Σ). The cycles of ecycle(G, Σ) are exactly the even cycles of signed graph (G, Σ). Moreover, the circuits of ecycle(G, Σ) come in two types:

1. $C \subseteq E$ such that C is an even circuit of (G, Σ), and,

2. $C_1 \cup C_2 \subseteq E$ such that C_1 and C_2 are both odd circuits of (G, Σ) and share at most one vertex.

Let D denote the dual of ecycle(G, Σ). The cycle space of D is spanned by the row space of A'. The rows of A' are the characteristic vectors of cuts of G and the characteristic vector of Σ. For signed graph (G, Σ): the symmetric difference of two cuts is a cut, the symmetric difference of two signatures is a cut, and the symmetric difference of a signature and a cut is a signature. It follows that the cycles of the D are exactly the cuts and signatures of (G, Σ).

2.3.3   Even Cut Matroids

Even cut matroids were also introduced by Zaslavsky [35] and have a nice interpretation with respect to grafts. Given graph G, let A be a matrix representation of $M^*(G)$. Let A' be obtained by augmenting A with row x^T outside the row space of A where x^T is the characteristic vector of $\Sigma \subseteq E$. Let T be the set of odd degree vertices in $G[\Sigma]$. Note that $|T|$ is even since any graph has an even number of odd degree vertices. The matroid with matrix representation A' is denoted by ecut(G, T) and called the even cut matroid of (G, T). The cycles of ecut(G, T) are exactly the *T-even cuts* of G; a cut $\delta(U)$ of G is T-even if $|U \cap T|$ is even.

Let D denote the dual of ecut(G, T). The cycle space of D is spanned by the row space of A'. The rows of A' are the characteristic vectors of cycles of G and the characteristic vector of Σ. A *T-join* is $F \subseteq E(G)$ such that the odd degree vertices of $G[F]$ are exactly T; since T is the set of odd vertices of $G[\Sigma]$, Σ is a T-join. In graft (G, T): the symmetric difference of two cycles is a cycle, the symmetric difference of two T-joins is a cycle, and the symmetric difference of a cycle and a T-join is a T-join. It follows that the cycles of the dual of ecut(G, T) are exactly the cycles and T-joins of (G, T).

2.4   Single Commodity Flow Results

Here we discuss Seymour's Theorem (Theorem 2.2.3) and the single commodity Flowing and Cycling Conjectures (Conjectures 2.2.4 and 2.2.5) for lifts and projections of graphic and cographic matroids.

2.4.1 Even Cycle Matroids

Let $M = \text{ecycle}(G, \Sigma)$ for graph $G = (V, E)$ and $\Sigma \subseteq E$. For $e = st \in \Sigma$, an e-path is either an odd st-path or the union of an even st-path and odd circuit sharing at most one vertex. The e-paths of an even cycle matroid are called *odd-st-walks*. If $s = t$, there are no odd-st-paths and all even st-paths have length zero; the e-paths of M are exactly the odd circuits of (G, Σ). Odd circuits have been studied extensively with respect to flows; see the next section (Section 2.4.2) for an overview of results. The single commodity Flowing Conjecture (Conjecture 2.2.4) is true for even cycle matroids [19], but the single commodity Cycling Conjecture (Conjecture 2.2.5) remains open.

Theorem 2.4.1. *If M is an even cycle matroid, then $(M, \{e\})$ has the fractional MFMC property if and only if it has no $(T_{11}^*, \{e_{11}\})$ or $(AG(3, 2), \{e_8\})$ minor.*

2.4.2 Even Cycle Matroids: e a loop

The single commodity Flowing and Cycling Conjectures (Conjectures 2.2.4 and 2.2.5) are resolved for even cycle matroids whenever e is a loop of G [20] [13].

Theorem 2.4.2. *If M is an even cycle matroid and e is a loop of underlying graph G, then $(M, \{e\})$ has the fractional MFMC property if and only if it has no $(T_{11}^*, \{e_{11}\})$ minor.*

Theorem 2.4.3. *If M is an even cycle matroid and e is a loop of underlying graph G, then $(M, \{e\})$ has the Eulerian integer MFMC property if and only if it has no $(T_{11}^*, \{e_{11}\})$ minor.*

Although it may seem unintuitive to consider the case when e is a loop, this case actually corresponds exactly to multi-commodity flow in graphs. Given (G, Σ) and $w \in \mathbb{Z}_+^E$, let H be obtained from G by adding loop $e \in \Sigma$ and let $M = \text{ecycle}(H, \Sigma)$. The e-paths of M are exactly the odd circuits of (G, Σ). Finding a packing of e-paths is equivalent to solving the multi-commodity flow instance corresponding to (G, Σ) as per Remark 2.2.2.

Signed graph (G, Σ) has no odd-K_5 minor if and only if $(M, \{e\})$ has no $(T_{11}^*, \{e_{11}\})$ minor. Minor operations in $(M, \{e\})$ can be shown to correspond to minor operations in (G, Σ). Thus Theorem 2.4.2 proves Theorem 1.1.4 regarding Eulerian integer multi-commodity flow for graphs and Theorem 2.4.3 proves Theorem 1.1.3 regarding fractional multi-commodity flow for graphs.

2.4.3 Even Cut Matroids

Let $M = \text{ecut}(G, T)$ for graph $G = (V, E)$ and $T \subseteq V$ with $|T|$ even. For $e = st \in \Sigma$, e-paths of M are the s, t-separating T-odd cuts of G. We say $\delta(S)$ as an s, t-separating T-odd cut if $s \in S$, $t \notin S$ and $|T \cap S|$ is odd. The e-paths of an even cut matroid are st-T-cuts. If e is a pendent edge ($e = \delta(v)$ for some $v \in V$), then the e-paths of M are simply the T-cuts of (G, T).

The single commodity Flowing Conjecture (Conjecture 2.2.4) is true for even cut matroids, but the single commodity Cycling Conjecture (Conjecture 2.2.5) remains open.

Theorem 2.4.4. *If M is an even cut matroid, then $(M, \{e\})$ has the fractional MFMC property if and only if it has no $(T_{11}, \{e_{11}\})$ or $(AG(3, 2), \{e_8\})$ minor.*

2.4.4 Duals of Even Cycle Matroids

Recall that if $M = \text{ecycle}(G, \Sigma)$ for $G = (V, E)$ and $\Sigma \subseteq E$, then the elements of M^* are cuts and signatures of (G, Σ). The e-paths of M^* are the st-bonds and signatures of (G, Σ). The single commodity Flowing Conjecture (Conjecture 2.2.4) holds for duals of even cycle matroids (follows from [19] and relationship between fractional flow in matroids and their duals - see Theorem 3.1.1).

Theorem 2.4.5. *If M^* is the dual of even cycle matroid M, then $(M^*, \{e\})$ has the fractional MFMC property if and only it has no $(T_{11}, \{e_{11}\})$ or $(AG(3, 2), \{e_8\})$ minor.*

The single commodity Cycling Conjecture (Conjecture 2.2.5) is still open for duals of even cycle matroids. However, it has been shown to imply the Four Colour Theorem: "every planar graph is 4-(vertex)-colourable".

Claim 2.4.6. *The single commodity Cycling Conjecture (Conjecture 2.2.5) for duals of even cycle matroids implies the Four Colour Theorem.*

Proof. Let $G = (V, E)$ be a simple plane graph, H be obtained from G by adding loop e, and $\Sigma = E(H)$. Let M be the dual of $\text{ecycle}(H, \Sigma)$. First, we describe the e-paths and e-cuts of M.

The cycles of M are signatures of (H, Σ) or cuts of H. Since no cut contains a loop, the e-paths of M are the signatures of $(G, E(G))$. That is, the e-paths of M are exactly the odd circuit covers of G.

Since the e-cuts of M are the e-paths of M^*, the e-cuts of M are of the form $D - e$ where D is a cocyle of M using e. In particular, D is an even cycle of M. Since e is an odd loop, the e-cuts of M are exactly the odd cycles of G.

Consider the unit weight function. The Eulerian condition holds if $|D|$ is even for all cocycles of M. Since the cocycles of M are the even cycles of (H, Σ) and $\Sigma = E(H)$, the unit weight function is Eulerian.

If the single commodity Cycling Conjecture holds, and (H, Σ) has no obstruction, then the size of a minimum length odd circuit is equal to the maximum number of disjoint signatures. Since G is simple, the length of the shortest odd circuit is at least 3 and so there exists disjoint signatures Σ_1 and Σ_2.

Suppose Σ_1 is the complement of the cut $\delta(A)$ and Σ_2 is the complement of the cut $\delta(B)$. Partition the vertices into sets $V_1 = A \cap B$, $V_2 = A \cap \bar{B}$, $V_3 = \bar{A} \cap B$ and $V_4 = \bar{A} \cap \bar{B}$. If V_1, V_2, V_3 and V_4 are stable sets of G, G is 4-colourable. By way of contradiction, suppose there exists edge $e = xy$ such that $x, y \in V_i$. Then $e \in \Sigma_1$ and $e \in \Sigma_2$, contradicting the fact Σ_1 and Σ_2 are disjoint. So (V_1, V_2, V_3, V_4) give a 4-colouring of $V(G)$.

If G is planar, it has no K_5 minor (Kuratowski's Theorem) and so (H, Σ) has no obstruction. It follows that G has a 4-colouring. Therefore, assuming the single commodity Cycling Conjecture for duals of even cycle matroids, one can prove the Four Colour Theorem. □

2.4.5   Duals of Even Cut Matroids

Recall that if $M = \text{ecut}(G, T)$ for $G = (V, E)$ and $T \subseteq V$, then the elements of M^* are cycles of G and T-joins of (G, T). The e-paths of M^* are the st-paths of G and T-join of (G, T). The single commodity Flowing Conjecture (Conjecture 2.2.4) holds for duals of even cut matroids (follows from [19] and relationship between fractional flow in matroids and their duals - see Theorem 3.1.1).

Theorem 2.4.7. *If M^* is the dual of even cut matroid M, then $(M^*, \{e\})$ has the fractional MFMC property if and only if it has no $(T_{11}^*, \{e_{11}\})$ or $(AG(3, 2), \{e_8\})$ minor.*

The single commodity Cycling Conjecture (Conjecture 2.2.5) is still open for duals of even cut matroids. However, it too has been shown to imply the Four Colour Theorem.

Claim 2.4.8. *The single commodity Cycling Conjecture (Conjecture 2.2.5) for duals of even cut matroids implies the Four Colour Theorem.*

Proof. First note a result of Tait [32] shows that the Four Colour Theorem is equivalent to the statement: "Every simple 2-connected cubic planar graph G is 3-edge colourable." Let G be a simple 2-connected cubic graph. The unit weight function is Eulerian and so the Cycling Conjecture (Conjecture 2.2.5) implies that G has 3 disjoint T-joins as long as G is not contractible to the Petersen graph. Since all vertices have degree 3, this is equivalent to being able to find 3 disjoint matchings. Assigning one colour class to each matching gives a 3-edge colouring of G. The Petersen graph is not planar, implying the Four Colour Theorem. □

It was, in fact, a conjecture of Tutte [34] that every simple 2-connected cubic graph that is not 3-edge-colourable is contractible to the Petersen graph. This conjecture was resolved by Robertson, Sanders, Seymour and Thomas [27] [26]. Tutte's conjecture has been generalized by Conforti and Johnson [5]; the more general conjecture would also follow from the single commodity Cycling Conjecture (Conjecture 2.2.5) for duals of even cut matroids.

Conjecture 2.4.9 (Conforti Johnson Conjecture). *In graphs not contractible to the Petersen graph, the number of pairwise disjoint postman sets is equal to the size of a minimum odd cut.*

2.4.6   A Note on the Restriction to Binary Matroids

Although we have restricted ourselves to binary matroids, all the flow definitions can be applied to general matroids. It has been noted previously [29] that the fractional, integer and Eulerian integer flow problems in non-binary matroids are (if you are a pessimist) uninteresting or (if you are an optimist) already solved. The matroid U_4^2 has a 4 element ground set and every 2 element subset thereof is independent. For any element f, there is no fractional w-flow for instance (U_4^2, f, w) where $w_e = 1$ for all $e \in E - f$ and $w_f = 2$. A result of Bixby [3] states that if M is connected and non-binary, then M has a U_4^2 minor using f. Thus single commodity flow instances for non-binary matroids can be disregarded.

2.5   Relating Flows to Clutters

2.5.1   Definitions

A *clutter* $\mathcal{F}$ on ground set E is a finite family of sets of $E := E(\mathcal{F})$ such that no set in $\mathcal{F}$ contains or is equal to some other set of $\mathcal{F}$. That is, there does not exist distinct

$F_1, F_2 \in \mathcal{F}$ such that $F_1 \subseteq F_2$. A set $B \subseteq E$ is a *cover* of $\mathcal{F}$ if $B \cap S \neq \emptyset$ for all $S \in \mathcal{F}$. The set of all inclusion-wise minimal covers of $\mathcal{F}$ forms a clutter $b(\mathcal{F})$ called the *blocker* of $\mathcal{F}$. As $b(b(\mathcal{F})) = \mathcal{F}$ [6], we call a pair of clutters $(\mathcal{F}, \mathcal{K})$ a *blocking pair* if $\mathcal{K} = b(\mathcal{F})$. A clutter $\mathcal{F}$ is *binary* if for all $S_1, S_2, S_3 \in \mathcal{F}$ there exists $S \in \mathcal{F}$ such that $S \subseteq S_1 \triangle S_2 \triangle S_3$. Equivalently, $\mathcal{F}$ is binary if for all $S \in \mathcal{F}$ and $T \in b(\mathcal{F})$, $|S \cap T|$ is odd [6].

The following results relate clutters and matroids and allow us to switch from discussing flows in matroids to discussing packings in clutters. Proofs appear in past work; see [19] for example.

Proposition 2.5.1. *Let M be a binary matroid and let $e \in E(M)$.*

1. *The set of e-paths of M is a binary clutter.*
2. *The set of e-cuts of M is a binary clutter.*
3. *The clutter of e-paths and e-cuts form a blocking pair.*

Moreover, every binary clutter is the set of e-paths for some binary matroid M and some $e \in E(M)$.

Remark 2.5.2. *The set of e-cuts of binary matroid M is exactly the set of e-paths of M^*.*

2.5.2 The MFMC Properties

Given clutter $\mathcal{F}$ with ground set E and $w \in \mathbb{Z}_+^E$, call $\mathcal{P} \subseteq \mathcal{F}$ a *w-packing* of elements of $\mathcal{F}$ if $|P \in \mathcal{P} : f \in P| \leq w_f$ for all $f \in E$. The size of $\mathcal{P}$ is simply $|\mathcal{P}|$. Consider the following primal-dual pair of linear programs.

$$
\begin{aligned}
\min \quad & \sum(w_f x_f : f \in E) \\
\text{subject to} \quad & \sum(x_f : f \in S) \geq 1 && \text{for all } S \in \mathcal{F} && (2.5) \\
& x \geq \mathbb{0}
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & \sum(y_S : S \in \mathcal{F}) \\
\text{subject to} \quad & \sum(y_S : f \in S \in \mathcal{F}) \leq w_f && \text{for all } f \in E && (2.6) \\
& y \geq \mathbb{0}.
\end{aligned}
$$

Observe that if $\mathcal{F}$ is the clutter of e-paths for binary matroid M and $e \in E(M)$, then (2.5) is the same as (2.3) and (2.6) is the same as (2.4). The restriction of (2.5) to the integers

models the problem of finding the minimum weight cover of $\mathcal{F}$ (a minimum weight element of $b(\mathcal{F})$). The restriction of (2.6) to the integers models the problem of finding maximum weight w-packing of elements of $\mathcal{F}$. In correspondence with our definitions for single commodity flow, we say that $\mathcal{F}$ has the

1. *fractional MFMC property* if for every $w \in \mathbb{Z}_+^E$ there is an integer solution to (2.5) and a solution to (2.6) with the same objective value,
2. *integer MFMC property* if for every $w \in \mathbb{Z}_+^E$ there is an integer solution to (2.5) and an integer solution to (2.6) with the same objective value,
3. *Eulerian integer MFMC property* if for every Eulerian $w \in \mathbb{Z}_+^E$ there is an integer solution to (2.5) and an integer solution to (2.6) with the same objective value.

By equivalence of definition: $(M, \{e\})$ has the fractional MFMC property if and only if the clutter of e-paths of M has the fractional MFMC property; $(M, \{e\})$ has the integer MFMC property if and only if the clutter of e-paths of M has the integer MFMC property; and $(M, \{e\})$ has the Eulerian integer MFMC property if and only if the clutter of e-paths of M has the Eulerian integer MFMC property. That is, instead of looking for single commodity w-flows, we can concern ourself with w-packings of clutters of e-paths.

2.5.3 Flowing and Cycling Conjectures for Clutters

Given clutter $\mathcal{F}$ and $f \in E(\mathcal{F})$, we define $\mathcal{F} \setminus f$ as $\{S \in \mathcal{F} : f \notin S\}$ and say $\mathcal{F} \setminus f$ is obtained from $\mathcal{F}$ by *deleting f*. Given clutter $\mathcal{F}$ and $f \in E(\mathcal{F})$, we define $\mathcal{F}/f$ as the set of inclusion-wise minimal sets of $\{S - f : S \in \mathcal{F}\}$ and say $\mathcal{F}/f$ is obtained from $\mathcal{F}$ by *contracting f*. A *minor* of $\mathcal{F}$ is any clutter $\mathcal{F}'$ obtained by a sequence of deletions and contractions. As deletions and contractions associate, we can unambiguously write $\mathcal{F}' = \mathcal{F}/I \setminus J$ to indicate that the elements of $I \subseteq E(\mathcal{F})$ were deleted and the elements of $J \subseteq E(\mathcal{F})$ were contracted to obtain $\mathcal{F}'$ from $\mathcal{F}$. It is straightforward to verify that any minor of a binary clutter is binary. Moreover, it is straightforward to show that if $\mathcal{F}$ is the clutter of e-paths of M, then $\mathcal{F} \setminus f$ is the clutter of e-paths of $M \setminus f$ and $\mathcal{F}/f$ is the clutter of e-paths of M/f.

The clutter $\mathcal{Q}_6$ is the clutter of odd circuits of K_4, the clutter $\mathcal{O}_{K_5}$ is the clutter of odd circuits of K_5, and the clutter $\mathcal{L}_7$ is the clutter of lines of the Fano plane. Two clutters are *isomorphic* to each other if one can be obtained from the other by relabelling elements of the ground set. Seymour's Theorem (Theorem 1.2.2) and the Flowing and Cycling Conjectures (Conjectures 1.2.3 and 1.2.4) can be restated as follows.

Theorem 2.5.3 (Seymour). *A binary clutter has the integer MFMC property if and only if it does not have a minor isomorphic to*

$$\mathcal{Q}_6 = \big\{\{1,2,4\},\{1,3,5\},\{2,3,6\},\{4,5,6\}\big\} \ .$$

Conjecture 2.5.4 (Flowing Conjecture). *A binary clutter is has the fractional MFMC property if and only if it does not have any of the following clutters as a minor: $\mathcal{O}_{K_5}$, $b(\mathcal{O}_{K_5})$, or $\mathcal{L}_7$.*

Conjecture 2.5.5 (Cycling Conjecture). *A binary clutter has the Eulerian MFMC property if and only if it does not have any of the following clutters as a minor: $\mathcal{O}_{K_5}$, $b(\mathcal{O}_{K_5})$, $\mathcal{L}_7$, or $\mathcal{P}_T$. The clutter $\mathcal{P}_T$ corresponds to the postman sets of the Petersen graph.*

2.5.4 Flowing and Cycling Results for Clutters

The following chart summarizes the implications of Section 2.4 to clutters.

	Integer	Fractional	Eulerian Integer
General	$\mathcal{Q}_6$	Flowing Conjecture	Cycling Conjecture
st-**Paths**	-	-	-
st-**Cuts**	-	-	-
Odd Circuits	$\mathcal{Q}_6$	$\mathcal{O}_{K_5}$	$\mathcal{O}_{K_5}$
Odd-*st*-Walks	$\mathcal{Q}_6$	$\mathcal{O}_{K_5}, \mathcal{L}_7$	Holds for non-rooted case [1]
st-*T*-**cuts**	$\mathcal{Q}_6$	$b(\mathcal{O}_{K_5}), \mathcal{L}_7$	Open
Blocker of Odd-*st*-Walks	$\mathcal{Q}_6$	$b(\mathcal{O}_{K_5}), \mathcal{L}_7$	Implies 4-Colour Theorem
Blocker of *st*-*T*-cuts	$\mathcal{Q}_6$	$\mathcal{O}_{K_5}, \mathcal{L}_7$	Implies Conforti Johnson Conjecture

Table 2.1: Whenever a minor or minors are listed, this indicates these minors are known to be the obstructions to the affiliated MFMC property. An entry of "-" implies there are no obstructions. Otherwise, the entry gives information about the related conjecture.

Chapter 3

A Constructive Version of Lehman's Theorem

3.1 Preliminaries

3.1.1 Set Covering Polyhedra, Idealness, and Lehman's Theorem

Given a $0,1$-matrix M we define the affiliated *set covering polyhedron* by

$$Q(M) := \{x \geq \mathbb{0} : Mx \geq \mathbb{1}\}.$$

We say M is *ideal* if $Q(M)$ is integral; i.e., if every extreme point of $Q(M)$ is integral. Given a clutter $\mathcal{F}$, let $M(\mathcal{F})$ denote the $0,1$-matrix with columns indexed by elements $f \in E(M)$ and rows indexed by sets $F \in \mathcal{F}$, where entry (F, f) is 1 if and only if $f \in F$. Observe that $M(\mathcal{F})$ is only defined up to permutations of the rows. We say clutter $\mathcal{F}$ is ideal if $M(\mathcal{F})$ is ideal, and write $Q(\mathcal{F})$ for $Q\big(M(\mathcal{F})\big)$. Note that if there exists $w \in \mathbb{Z}_+^E$ such that $\min\{w^T x : x \in Q(\mathcal{F})\}$ is fractional, then $\mathcal{F}$ is not ideal. Ideal clutters are exactly those clutters for which the fractional MFMC property holds. A theorem of Lehman [22] relates idealness in a clutter to idealness in its blocker.

Theorem 3.1.1. *A clutter is ideal if and only if its blocker is.*

Note that we have already explicitly and implicitly used this theorem. For example, it shows that $(M, \{e\})$ has the fractional MFMC property if and only if $(M^*, \{e\})$ has the fractional MFMC property.

We call clutter $\mathcal{F}$ *minimally nonideal (mni)* if it is not ideal but all its proper minors are ideal. An example of a mni clutter is the clutter whose ground set and elements correspond, respectively, to the points and lines of the finite degenerate projective plane. This clutter on ground set $\{0, 1 \ldots s\}$ is given by

$$\mathcal{J}_s = \big\{ \{1 \ldots s\}, \{0, 1\}, \{0, 2\}, \ldots, \{0, s\} \big\}$$

for $s \geq 2$.

A theorem of Lehman [23] gives a characterization of mni clutters and shows, surprisingly, that set covering polyhedra corresponding to minimally nonideal clutters have exactly one fractional extreme point.

Theorem 3.1.2 (Lehman's Theorem). *Let $\mathcal{F}$ be a minimally non-ideal clutter that is not isomorphic to $\mathcal{J}_s$ for $s \geq 2$. Let $\mathcal{K}$ be the blocker of $\mathcal{F}$. Denote by $\bar{\mathcal{F}}$ the clutter formed by the sets of minimum cardinality of $\mathcal{F}$. Denote by $\bar{\mathcal{K}}$ the clutter formed by the sets of minimum cardinality of $\mathcal{K}$. Then*

(L1) $M(\bar{\mathcal{F}})$ and $M(\bar{\mathcal{K}})$ are square matrices, and
(L2) after possibly rearranging rows of $M(\bar{\mathcal{F}})$ we have for some $d \geq 1$

$$M(\bar{\mathcal{F}})M(\bar{\mathcal{K}})^T = J + dI. \tag{3.1}$$

where J is the dimension n square matrix of 1s and I is the dimension n identity matrix.

In the preceding theorem, if r denotes the cardinality of the sets of $\bar{\mathcal{F}}$, then $\frac{1}{r}\mathbb{1}$ is the unique fractional extreme point of $Q(\mathcal{F})$. If ℓ denotes the cardinality of the sets of $\bar{\mathcal{K}}$, then $\frac{1}{\ell}\mathbb{1}$ is the unique fractional extreme point of $Q(\mathcal{K})$. The clutters $\mathcal{O}_{K_5}$, $b(\mathcal{O}_{K_5})$ and $\mathcal{L}_7$ are all minimally nonideal. In Section 3.4, an interpretation of equation (3.1) is given for clutter $\mathcal{O}_{K_5}$.

Let $(\mathcal{F}, \mathcal{K})$ be a blocking pair, $\bar{\mathcal{F}}$ denote the clutter formed by the minimum cardinality sets of $\mathcal{F}$, and $\bar{\mathcal{K}}$ denote the clutter formed by the minimum cardinality sets of $\mathcal{K}$. We say that $\mathcal{F}$ is a *Lehman clutter* whenever conditions (L1) and (L2) of Lehman's Theorem are satisfied. Note that if $\mathcal{F}$ is a Lehman clutter, so also is its blocker $\mathcal{K}$. Hence we refer to $(\mathcal{F}, \mathcal{K})$ as a *Lehman blocking pair*.

3.1.2 Statement of Constructive Version

Consider the decision problem: is a given clutter $\mathcal{F}$ ideal? It is clearly in Co-NP as it suffices to exhibit a fractional extreme point of $Q(\mathcal{F})$. Outcomes (1) and (2) of the previous

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theorem are also a Co-NP certificate. For any nonideal clutter $\mathcal{F}$, there exists a clutter minor of $\mathcal{F}$ that is a Lehman clutter. If $\mathcal{F}' = \mathcal{F}/I \setminus J$, then I, J, $M(\mathcal{F}')$ and $M(b(\mathcal{F}'))$ also give a certificate that $\mathcal{F}$ is nonideal. We are interested in finding such certificates efficiently. Firstly, doing so allows us to construct a highly regular Co-NP certificate of nonidealness from an arbitrary fractional extreme point. Moreover, combining this with existing excluded minor results allows us to find excluded minors for idealness.

Before stating the main theorem of this section, we note that the result relies upon the ellipsoid method. As is typical in algorithms using the ellipsoid method, the algorithm's runtime is analyzed with respect to the number of calls to a separation oracle. Let $P \subseteq \mathbb{R}^n$ be a polyhedron. The *separation problem* for P and a point $\bar{x} \in \mathbb{Q}^n$ is to either determine that $\bar{x} \in P$, or to find a separating constraint $a^T x \leq \beta$ (i.e., $a^T \bar{x} > \beta$ and $a^T x \leq \beta$ for all $x \in P$). A *separation oracle* for P is a function that solves the separation problem for any $\bar{x} \in \mathbb{Q}^n$. We are now ready to state our algorithmic version of Theorem 3.1.2.

Theorem 3.1.3. *Let $\mathcal{F}$ be a clutter and suppose that $Q(\mathcal{F}) \subseteq \mathbb{R}^n$ is given by a separation oracle. Let x be a fractional extreme point of $Q(\mathcal{F})$ and suppose that we are given n facets of $Q(\mathcal{F})$ that define x. Then in oracle polynomial time of n we can find disjoint sets $I, J \subseteq E(\mathcal{F})$ such that for $\mathcal{F}' = \mathcal{F}/I \setminus J$ either*

1. $\mathcal{F}'$ is isomorphic to $\mathcal{J}_s$, or
2. $\mathcal{F}'$ is a Lehman clutter.

Moreover, in case (2) we also find all the minimum cardinality sets of $\mathcal{F}'$.

3.2 Proof Outline

3.2.1 Key Definitions and Lemmas

Throughout this section, $\mathcal{F}$ will always denote a clutter with $E(\mathcal{F}) = [n]$[1]. Given a clutter $\mathcal{F}$, we denote by $P(\mathcal{F})$ the polytope $Q(\mathcal{F}) \cap [0,1]^n$.

Proposition 3.2.1. *Polyhedron $P(\mathcal{F})$ is integral if and only if $Q(\mathcal{F})$ is integral.*

Proof. Suppose $P(\mathcal{F})$ is integral and $\bar{x}$ is a fractional extreme point of $Q(\mathcal{F})$.

First consider the case when $\bar{x} \in P(\mathcal{F})$. Since $\bar{x}$ cannot be a fractional extreme point of $P(\mathcal{F})$, we can write $\bar{x} = \frac{1}{2}x^1 + \frac{1}{2}x^2$ for $x^1, x^2 \in P(\mathcal{F})$. Since $P(\mathcal{F}) \subset Q(\mathcal{F})$, $x^1, x^2 \in Q(\mathcal{F})$; this contradicts the fact $\bar{x}$ is an extreme point of $Q(\mathcal{F})$.

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[1]Here $[n]$ denotes the set $\{1, 2 \ldots n\}$.

Now, consider the case when $\bar{x} \notin P(\mathcal{F})$. There exists $i \in [n]$ such that $\bar{x}_i > 1$. Define x^1 by $x_j^1 = \bar{x}_j$ for all $j \neq i$ and $x_i^1 = \bar{x}_i - \epsilon$. Define x^2 by $x_j^2 = \bar{x}_j$ for all $j \neq i$ and $x_i^2 = \bar{x}_i + \epsilon$. Choosing ϵ small enough, $x^1, x^2 \in Q(\mathcal{F})$. Then $\frac{1}{2}x^1 + \frac{1}{2}x^2 = \bar{x}$, contradicting the fact $\bar{x}$ is an extreme point of $Q(\mathcal{F})$.

Suppose $Q(\mathcal{F})$ is integral and $\bar{x}$ is a fractional extreme point of $P(\mathcal{F})$.

If $x_i < 1$ for all i, then $\bar{x}$ is also a fractional extreme point of $Q(\mathcal{F})$. Otherwise we can write $\bar{x} = \frac{1}{2}x^1 + \frac{1}{2}x^2$ for $x^1 = \bar{x} + \epsilon d$ and $x^2 = \bar{x} - \epsilon d$. Choosing ϵ small enough, $x^1, x^2 \in P(\mathcal{F})$, contradicting the fact $\bar{x}$ is an extreme point of $P(\mathcal{F})$. However, concluding $\bar{x}$ is is a fractional extreme point of $Q(\mathcal{F})$ contradicts the fact $Q(\mathcal{F})$ is integral. Thus there exists $i \in [n]$ such that $i = 1$.

Project $P(\mathcal{F})$ and $Q(\mathcal{F})$ onto $\{x : x_i = 1\}$; that is, proceed inductively on $P(\mathcal{F}/i)$ and $Q(\mathcal{F}/i)$. Whenever $\mathcal{F}$ is a one element clutter, $P(\mathcal{F})$ is integral if and only if $Q(\mathcal{F})$ integral. It follows by induction that $P(\mathcal{F})$ is integral if and only if $Q(\mathcal{F})$ is integral. $\square$

Given a point $\bar{x} \in P(\mathcal{F})$ and $j \in [n]$, let $\bar{x}^j = (\bar{x}_1, \ldots, \bar{x}_{j-1}, 1, \bar{x}_{j+1}, \ldots, \bar{x}_n)^T$ and $F^j = P(\mathcal{F}) \cap \{x : x_j = 1\}$. Note that $\bar{x}^j \in F^j$. Call point $\bar{x} \in P(\mathcal{F})$ *special* if it is an extreme point and for all $j \in [n]$

(S1) $0 < \bar{x}_j < 1$,
(S2) $\bar{x}^j$ can be expressed as a convex combination of integer extreme points of F^j.

Our algorithm relies on the following lemma.

Lemma 3.2.2. *Suppose $P(\mathcal{F})$ is given by a separation oracle. Given an extreme point $\bar{x}$ of $P(\mathcal{F})$ with n facets that define $\bar{x}$, we can, in oracle polynomial time in n, either*

1. *deduce that $\bar{x}$ is special, or*
2. *find $j \in [n]$ and a fractional extreme point x' of $P(\mathcal{F}/j)$, or*
3. *find $j \in [n]$ and a fractional extreme point x' of $P(\mathcal{F} \setminus j)$.*

Moreover, for outcomes (2) and (3) we also find $n - 1$ facets that define x'.

Two extreme points in a polytope are *adjacent* if they are contained in a face of dimension 1. The set of all extreme points that are adjacent to extreme point $\bar{x}$ are the *neighbours* of $\bar{x}$. Extreme point $\bar{x}$ of polyhedron $P \subseteq \mathbb{R}^n$ is *non-degenerate* if there are exactly n facets of P satisfied at equality at $\bar{x}$ or, equivalently, if $\bar{x}$ has exactly n neighbours in P. These definitions are sufficient to state the next set of required lemmas.

Lemma 3.2.3. *Special points of $P(\mathcal{F})$ are non-degenerate.*

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Lemma 3.2.4. *Suppose we are given a non-degenerate extreme point $\bar{x}$ of $P(\mathcal{F})$ where $0 < \bar{x}_j < 1$ for all $j \in [n]$, and n facets that define $\bar{x}$. If $P(\mathcal{F})$ is described by a membership oracle, then we can, in oracle polynomial time of n, find the n neighbours of $\bar{x}$.*

Lemma 3.2.5. *No two special points of $P(\mathcal{F})$ are adjacent.*

Lemma 3.2.6. *Suppose that $\bar{x}$ is a special point of $P(\mathcal{F})$ and that all neighbours of $\bar{x}$ are integral. The facets that define $\bar{x}$ are of the form $\sum_{k \in S_i} x_k \geq 1$ where $S_i \in \mathcal{F}$ for $i = 1, \ldots, n$. Let $\bar{\mathcal{F}} = \{S_1, \ldots, S_n\}$; either*

1. $\mathcal{F}$ is isomorphic to $\mathcal{J}_s$ and $\bar{\mathcal{F}} = \mathcal{F}$, or
2. $\mathcal{F}$ is a Lehman clutter and the elements of $\bar{\mathcal{F}}$ are the minimum cardinality sets of $\mathcal{F}$.

Note that in Lemma 3.2.6, $\bar{x}$ is necessarily non-degenerate because of Lemma 3.2.3.

3.2.2 Algorithm

Let $\bar{x}$ be an extreme point of $P(\mathcal{F})$ and suppose that we are given n facets of $P(\mathcal{F})$ that define $\bar{x}$. We apply the algorithm referenced in Lemma 3.2.2. If outcome (2) or (3) occurs, then we apply the main algorithm recursively to $P(\mathcal{F}/i)$ or $P(\mathcal{F} \setminus i)$ respectively, with the new extreme point x' and the $n-1$ facets defining x'. Note that the separation oracle for $P(\mathcal{F})$ extends to a separation oracle for $P(\mathcal{F}/j)$ and $P(\mathcal{F} \setminus j)$, as contracting j corresponds to setting $\bar{x}_j = 0$ and deleting j to setting $\bar{x}_j = 1$.

Otherwise, outcome (1) of Lemma 3.2.2 occurs. Because of Lemma 3.2.3, $\bar{x}$ is non-degenerate. Using the algorithm referenced in Lemma 3.2.4 we can find its neighbours $b^1, \ldots, b^n$. Consider first the case where all of $b^1, \ldots, b^n$ are integer. Since we have the facets that define $\bar{x}$, we can construct $\bar{\mathcal{F}}$ as in Lemma 3.2.6. Then outcome (1) and (2) of Lemma 3.2.6 correspond respectively to outcomes (1) and (2) of Theorem 3.1.3 and we can stop.

Thus we may assume that b^i is fractional for some $i \in [n]$. We apply the algorithm referenced in Lemma 3.2.2 to b^i. As b^i is not special (see Lemma 3.2.5), outcome (2) or (3) occurs, and we can again apply the main algorithm recursively.

It is straightforward to verify that the algorithm runs in oracle polynomial time. Since $Q(\mathcal{F})$ is described by a separation oracle then, so is $P(\mathcal{F})$ as it suffices to check in addition that $x_e \leq 1$ for all $e \in E(\mathcal{F})$. Note that this algorithm may call itself recursively. Between recursive calls, we use the polynomial time algorithm referenced in Lemma 3.2.2 at most twice and the polynomial time algorithm referenced in Lemma 3.2.4 once. The polynomiality follows because every time we call the algorithm recursively we decrease the dimension of the polytope in consideration by one and so at most n recursive calls are made.

3.3 Technical Sections of Proof

3.3.1 Polyhedral Proofs

The following polyhedral theory definitions are found in Grötschel et. al. (see [16] Chapter 6.2). Let $P \subseteq \mathbb{R}^n$ be a polyhedron and let ϕ be positive integer. We say that P has *facet-complexity at most ϕ* if there exists a system of inequalities with rational coefficients that has solution set P where the encoding length of each inequality in the system is at most ϕ. (If $P \subseteq \mathbb{R}^n$, we require $\phi \geq n+1$.) A *well-described polyhedron* is a triple $(P; n, \phi)$ where $P \subseteq \mathbb{R}^n$ is a polyhedron with facet-complexity at most ϕ. The *encoding length* $\langle P \rangle$ of well-described polyhedron $(P; n, \phi)$ is at most $\phi + n$.

The proof of Lemma 3.2.2 requires the following algorithmic version of Caratheodory's Theorem (see [16] Theorem 6.5.11).

Proposition 3.3.1. *Let $P \subseteq \mathbb{R}^n$ be a well-described polytope given by a separation oracle and let $\bar{x} \in P \cap \mathbb{Q}^n$. There exists an oracle-polynomial algorithm that will express $\bar{x}$ as a convex combination of at most $\dim(P) + 1$ extreme points of P. Moreover, for each of these points we can find n defining facets of P.*

We will apply this proposition to show that given a separation oracle for $P(\mathcal{F})$, we can express any point $\bar{x} \in P(\mathcal{F}) \cap \mathbb{Q}^n$ as a convex combination of extreme points of $P(\mathcal{F})$. To do this, we show that $P(\mathcal{F})$ is indeed well-described. Since $P(\mathcal{F})$ can be described by a system of inequalities with $0, 1$ coefficients, the facet-complexity of $P(\mathcal{F})$ is at most $n + 1$ and so $(P(\mathcal{F}); n + 1, n)$ is a well-described polyhedron.

Corollary 3.3.2. *Let $P(\mathcal{F})$ be given by a separation oracle and let $\bar{x} \in P(\mathcal{F}) \cap \mathbb{Q}^n$. There exists an oracle-polynomial time algorithm that will express $\bar{x}$ as a convex combination of at most $\dim(P) + 1$ extreme points of $P(\mathcal{F})$. Moreover, for each of these points, we can find n defining facets of $P(\mathcal{F})$.*

Using this corollary, we are ready to prove Lemma 3.2.2.

Proof of Lemma 3.2.2. Suppose $\bar{x}_j \in \{0, 1\}$ for some $j \in [n]$. Let $x' = (\bar{x}_1, \ldots, \bar{x}_{j-1}, \bar{x}_{j+1}, \ldots, \bar{x}_n)^T$. If $\bar{x}_j = 0$ then x' is an extreme point of $P(\mathcal{F}/j)$ and the facets that define $\bar{x}$, omitting $x_j = 0$, define x' and outcome (2) of the lemma occurs. If $\bar{x}_j = 1$ then x' is an extreme point of $P(\mathcal{F} \setminus j)$ and the facets that define $\bar{x}$, omitting $x_j = 1$, define x' and outcome (3) of the lemma occurs. Thus we may assume condition (S1) holds.

For every $j \in [n]$, use Proposition 3.3.2 to express $\bar{x}^j$ as a convex combination of extreme points $y^1, \ldots, y^k$ of F^j where $k \leq n$. Note, that as $\bar{x}$ is an extreme point of $P(\mathcal{F})$, it has encoding length $\langle \bar{x} \rangle$ polynomial in n (see [16] Theorem 6.2.4). Suppose y^s is fractional for some $s \in [k]$, then $(y_1^s, \ldots, y_{j-1}^s, y_{j+1}^s, \ldots, y_n^s)^T$ is an extreme point $P(\mathcal{F} \setminus j)$ and outcome (3) occurs. If this never occurs, condition (S2) holds and $\bar{x}$ is special. Note that even if outcome (2) or (3) occurs, this does not imply $\bar{x}$ is not special (only that we have not succeeded in demonstrating that it is special). $\qquad\square$

No additional definitions are needed to prove Lemma 3.2.4. We give a sketch of the proof here; a true proof would require introducing more machinery (Cramer's rule to bound denominators, for example).

Proof Sketch of Lemma 3.2.4. As $\bar{x}$ is non-degenerate, exactly n constraints of $M(\mathcal{F})\bar{x} \geq \mathbb{1}$ are tight for $\bar{x}$. Find $d \neq \mathbb{O}$ satisfying $n - 1$ of these constraints. Then a neighbour b of $\bar{x}$ is on the line $L = \{\bar{x} + \lambda d : \lambda \in \mathbb{R}\}$. Use the membership oracle to do a binary search to find $b \in L$. $\qquad\square$

3.3.2 Proofs using Property (P)

The remainder of the proofs follow closely the proof of Lehman's Theorem (Theorem 3.1.2) appearing in [6]. The following definition will be key: An $n \times n$, $0, 1$-matrix M has *property (P)* if

(P1) M has no dominated row; that is, there do not exist rows R_i, $_j$ such that $\text{supp}(R_i) \subseteq \text{supp}(R_j)$,

(P2) M has no column of 1s,

(P3) for all $i, j \in [n]$ where $M_{i,j} = 0$,

$$\sum_{k=1}^{n} M_{ik} = \sum_{k=1}^{n} M_{jk}.$$

One key observation regarding matrices with property (P) is that they cannot differ in exactly one row. The argument we give for this fact essentially appears in [6] Lemma 4.11.

Lemma 3.3.3. *No pair of matrices with property (P) differ in exactly one row.*

Proof. Suppose for a contradiction that we have $n \times n$ matrices M, M' with property (P) where M has rows $R_1, \ldots, R_n$ and M' has rows $R_2, \ldots, R_n, R_{n+1}$. Property (P1) shows

there exists $j \in [n]$ such that entry j of R_1 is 0 and entry j of R_{n+1} is 1. Property (P2) states that column j of M' is not a column of 1s, and so for some i, $2 \leq i \leq n$, entry j of R_i is 0. By property (P3), the number of 1s in R_i is equal to both the number of 1s in column j of M and M'. This is a contradiction as column j of M' has one more 1 than column j of M. □

Let $\bar{x}$ be an extreme point of $P(\mathcal{F})$ where $0 < \bar{x}_j < 1$ for all $j \in [n]$. We say that $\bar{\mathcal{F}} \subseteq \mathcal{F}$ is a *core* of $\bar{x}$ if $M(\bar{\mathcal{F}})\bar{x} = \mathbb{1}$, and $M(\bar{\mathcal{F}})$ is square and non-singular. Clearly, if $\bar{x}$ is an extreme point then there exists a core for $\bar{x}$. Moreover, the core is unique if and only if $\bar{x}$ is non-degenerate. We aim to show $M(\bar{\mathcal{F}})$ has property (P) whenever $\bar{\mathcal{F}}$ is a core of a special point. We first require the following lemma due to de Bruijn and Erdös [8].

Lemma 3.3.4. *Let $(I \cup J, E)$ be a bipartite graph with no isolated node. If $|I| \geq |J|$ and $d(i) \geq d(j)$ for all $i \in I$, $j \in J$ such that $ij \in E$, then $|I| = |J|$ and $d(i) = d(j)$ for all $i \in I$, $j \in J$ such that $ij \in E$.*

Proof. Applying the given inequality and rearranging the summation gives

$$|I| = \sum_{i \in I} \left(\sum_{j \in N(i)} \frac{1}{\deg_G(i)} \right) \leq \sum_{i \in I} \left(\sum_{j \in N(i)} \frac{1}{\deg_G(j)} \right) = \sum_{j \in J} \left(\sum_{i \in N(j)} \frac{1}{\deg_G(j)} \right) = |J|$$

Since $|I| \geq |J|$, equality holds throughout and thus $|I| = |J|$ and $d(i) = d(j)$ for all $i \in I$, $j \in J$ such that $ij \in E$. □

Lemma 3.3.5. *Let $\bar{x}$ be a special point of $P(\mathcal{F})$ and let $\bar{\mathcal{F}}$ be a core of $\bar{x}$. Then $M(\bar{\mathcal{F}})$ has property (P).*

Proof. Matrix $M(\bar{\mathcal{F}})$ clearly satisfies (P1) as $\mathcal{F}$ is a clutter. Matrix $M(\bar{\mathcal{F}})$ has property (P2) as $M(\bar{\mathcal{F}})$ is nonsingular because $\bar{\mathcal{F}}$ is a core of $\bar{x}$ with special property (S1). Thus it suffices to show that (P3) holds for $M(\bar{\mathcal{F}})$. The proof is nearly identical to that of Lemma 4.10 in [6] for minimally nonideal clutters. The property of being special ensures $M(\bar{\mathcal{F}})$ has the same necessary properties implied by the assumption $\mathcal{F}$ is minimally nonideal.

Consider x^j defined previously by $x_j^j = 1$ and $x_k^j = x_k$ for $k \neq j$. Clearly $x^j \in P(\mathcal{F})$ since $P(\mathcal{F}) = \{x \in [0,1]^n : M(\mathcal{F}) \geq \mathbb{1}\}$. Let F_j (not necessarily equal to F^j defined previously) be the lowest dimensional face of $P(\mathcal{F}) \cap \{x : x_j = 1\}$ that contains x^j. Unlike in the minimally nonideal case, F_j is not necessarily integral. However, the result still follows by bounding the dimension of F_j two different ways.

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<u>Approach 1:</u> The point x^j lies at the intersection of the hyperplanes $\{x : x_j = 1\}$ and $\{x : (a^k)^T x = 1\}$ for all a^k rows of $M(\bar{\mathcal{F}})$ such that $a_{kj} = 0$ There are at least $n - \sum_{k=1}^{n} a_{kj}$ equations defining hyperplanes of the second type and the hyperplane of the first type is independent of these. Thus

$$\dim(F_j) \leq n - (n - \sum_{k=1}^{n} a_{kj} + 1) = \sum_{k=1}^{n} a_{kj} - 1. \tag{3.2}$$

<u>Approach 2:</u> Suppose that $(a^i)^T$ is such that $a_{ij} = 0$. Since $\bar{x}$ is special, there exist integral $b^1 \ldots b^l$ for $l \leq n$ such that $x^j = \sum_{t=1}^{l} \gamma_t b^t$ where $\gamma \geq 0$, $\mathbb{1}^T \gamma = 1$. Since each b^t is a $\{0, 1\}$ vector

$$1 = (a^i)^T x^i \qquad \text{since } a_{ij} = 0 \text{ and so such a constraint is also tight for } x^j$$

$$= \sum_{t=1}^{l} \gamma_l (a^i)^T b^t \qquad \text{substituting for } x^j$$

$$\geq 1 \qquad \text{since } (a^i)^T b^t \geq 1 \text{ as } b^t \in P(\mathcal{F}).$$

So, equality holds throughout. This implies that $(a^i)^T b^t = 1$ for all $t \in \{1 \ldots l\}$ and thus b^t has exactly one nonzero entry in every column k such that $a_{jk} = 1$. Furthermore, because $x_k^j = \sum_{t=1}^{l} \gamma_t b_k^j$ for all k such that $a_{ik} = 1$ and $x_k^j > 0$ for all k, F_j contains at least $\sum_{k=1}^{n} a_{ik}$ linearly independent points. Therefore,

$$\dim(F_j) \geq \sum_{k=1}^{n} a_{ik} - 1. \tag{3.3}$$

Combining bounds (3.2) and (3.3) regarding $\dim(F_j)$, we conclude that $\sum_{k=1}^{n} a_{ik} \leq \sum_{k=1}^{n} a_{kj}$ for all $i, j \in [n]$ such that $a_{ij} = 0$. Consider the bipartite representation of $J - M(\bar{\mathcal{F}})$ given by graph $G = (V, E)$ with bipartition (R, C) where $R = \{1, \ldots n\}$ corresponds to the rows of $M(\bar{\mathcal{F}})$ and $C = \{1', \ldots n'\}$ corresponds to the columns of $M(\bar{\mathcal{F}})$. For $i \in \{1 \ldots n\}$ and $j \in \{1', \ldots n'\}$, $ij \in E$ if and only if $(J - A)_{ij} = 1$. For $i \in \{1, \ldots n\}$, the degree of i is $\deg_G(i) = n - \sum_{k=1}^{n} a_{ik}$. For $j \in \{1' \ldots n'\}$, the degree of j is $\deg_G(j) = n - \sum_{k=1}^{n} a_{kj}$. If ij is an edge of the bipartite representation of $J - A$, then $a_{ij} = 0$ and so

$$\deg_G(i) = n - \sum_{k=1}^{n} a_{ik} \geq n - \sum_{k=1}^{n} a_{kj} = \deg_G(j)$$

Having shown all the conditions of Lemma 3.3.4 hold, we conclude that $\deg_G(i) = \deg_G(j)$ for all $i, j \in [n]$ such that $a_{ij} = 0$. Thus row i and column j of $M(\bar{\mathcal{F}})$ have the same number of 1s. That is, property (P3) holds. □

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We are now ready prove two more of the key lemmas introduced in Section 3.2.1.

Proof of Lemma 3.2.3. Suppose for a contradiction that special point $\bar{x}$ of $P(\mathcal{F})$ is degenerate. Then $\bar{x}$ must have two distinct cores $M(\bar{\mathcal{F}})$ and $M(\bar{\mathcal{F}}')$ that differ in exactly one row. Lemma 3.3.5 implies that these cores have property (P), contradicting Lemma 3.3.3. $\quad\square$

Lemma 3.2.5. Suppose for a contradiction there exists special points x' and x'' that are adjacent. By Lemma 3.2.3, x' has a unique core $\mathcal{F}'$ and x'' has a unique core $\mathcal{F}''$. Since x' and x'' are adjacent, $M(\mathcal{F}')$ and $M(\mathcal{F}'')$ differ in exactly one row. Lemma 3.3.5 implies that these cores have property (P), contradicting Lemma 3.3.3. $\quad\square$

Having proved all the other lemmas of Section 3.2.1, it remains to prove Lemma 3.2.6.

3.3.3 Proof of Lemma 3.2.6

Two matrices are *isomorphic* to one another if one can be obtained from the other by permuting rows and columns. Lemma 4.12 in the proof of Lehman's Theorem we are following [6] is equivalent to the following lemma.

Lemma 3.3.6. *Let Y and Z be $n \times n$ matrices. Suppose Y has property (P), and suppose that*

$$YZ^T = J + D$$

where D is a diagonal matrix with positive diagonal entries. Then either,

1. *Y and Z are both isomorphic to $M(\mathcal{J}_s)$ for $s \geq 2$, or*
2. *$D = dI$, where d is a positive integer.*

A $0,1$-matrix M is *r-regular*, if every row and every columns has exactly r 1s. The following result is due to Bridges and Ryser [4].

Theorem 3.3.7. *Let Y and Z be $n \times n$, $0,1$-matrices such that $YZ^T = J + dI$ for $d \geq 1$. Then Y is r-regular, Z is ℓ-regular where $r, \ell \geq 2$, $n = r\ell - d$, and $YZ^T = Z^TY$.*

Here, we differ from proof the of Lehman's Theorem we are following [6], only in that, we need a different argument to prove there are exactly n minimum cardinality sets in $b(\mathcal{F})$.

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Proof Lemma 3.2.6. Lemma 3.2.3 implies that $\bar{\mathcal{F}} = \{S_1, \ldots, S_n\}$ is the unique core of $\bar{x}$. Denote by $b^1, \ldots, b^n$ the (integer) neighbours of $\bar{x}$ and let Y be the $n \times n$ $0, 1$-matrix for which row i is the characteristic vector of S_i. For any $i \in [n]$, b^i and $\bar{x}$ are in a face of $P(\mathcal{F})$ of dimension 1 and so we may assume that the unique constraint of $Yx \geq \mathbb{1}$ that is not tight for b^i is constraint i. Hence, if we denote by Z the matrix with rows $b^1, \ldots, b^n$,

$$YZ^T = J + D,$$

where D is a diagonal matrix with positive diagonal entries.

Lemma 3.3.5 implies that Y has property (P). Together with Theorem 3.3.6, it implies that either Y is isomorphic to $M(\mathcal{J})$ or $D = dI$ for some integer $d \geq 1$. We may assume that the former case does not occur for otherwise we have the required outcome (1) of Lemma 3.2.6. By Theorem 3.3.7, Y is r-regular, Z is ℓ-regular for $r, \ell \geq 2$. It follows that $\bar{x} = \frac{1}{r}\mathbb{1}$ and since $\bar{x}$ has a unique core, the clutter $\bar{\mathcal{F}}$ is exactly the minimum cardinality sets of $\mathcal{F}$.

Since the neighbours of $\bar{x}$ are $b^1, \ldots, b^n$ and since $b^1, \ldots, b^n$ have all exactly ℓ 1s, the constraint $\sum_{j=1}^{n} x_j \geq \ell$ is valid for all $0, 1$ points of $P(\mathcal{F})$. It follows, in particular, that every element in the blocker $\mathcal{K}$ of $\mathcal{F}$ has cardinality at least ℓ. Let $\bar{\mathcal{K}}$ denote the minimum cardinality sets of $\mathcal{K}$. Then $\bar{\mathcal{K}}$ contains all sets corresponding to $b^1, \ldots, b^n$.

Suppose for a contradiction there exists a set $T \in \bar{\mathcal{K}}$ that does not correspond to any of $b^1, \ldots, b^k$ and let b denote the characteristic vector of T. Since $T \in b(\mathcal{F})$, b is an extreme point of $P(\mathcal{F})$ (see Remark 1.16 in [6]). Since b is a linear combination of $b^1, \ldots, b^n$ (because $\{b^1, \ldots b^n\}$ has rank n), there exists vector α such that

$$\sum_{j=1}^{n} \alpha_j b^j = b. \tag{3.4}$$

By right multiplying the previous equation on both sides by $\frac{1}{\ell}\mathbb{1}$ we obtain

$$\sum_{j=1}^{n} \alpha_j = 1. \tag{3.5}$$

Combining (3.4) and (3.5), we get

$$\sum_{j=1}^{n} \alpha_j(b^j - \bar{x}) = b - \bar{x}. \tag{3.6}$$

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Since $b^1, \ldots, b^n$ are neighbours of $\bar{x}$ and since $b \in P(\mathcal{F})$, $b - \bar{x}$ is in the cone formed by the vectors $b^1 - \bar{x}, \ldots, b^n - \bar{x}$, i.e. for some $\lambda \geq 0$

$$\sum_{j=1}^{n} \lambda_j (b^j - \bar{x}) = b - \bar{x}. \qquad (3.7)$$

Subtracting (3.7) from (3.6) yields

$$\sum_{j=1}^{n} (\alpha_j - \lambda_j)(b^j - \bar{x}) = 0. \qquad (3.8)$$

Since $P(\mathcal{F})$ is full dimensional, $b^1 - \bar{x}, \ldots, b^n - \bar{x}$ are linearly independent. It follows from (3.8) that $\alpha = \lambda$ and in particular, $\alpha \geq 0$. Combining this with (3.4) and (3.5) shows b is a convex combination of $b^1, \ldots, b^n$, contradicting the fact b is an extreme point of $P(\mathcal{F})$. $\qquad \square$

3.4   Exploring Lehman's Theorem

Lehman's Theorem admits many combinatorial interpretations. In this section, an interpretation of the matrix equation $M(\bar{\mathcal{F}})M(\bar{\mathcal{K}})^T = J + dI$ for two mni clutters is provided. Some of the many useful corollaries of Lehman's Theorem are stated and proven. All of the corollaries appearing here have been noted previously (see [18], [19], [28], for example).

3.4.1   Interpreting Lehman's Theorem: $\mathcal{C}_{2,3}$ and $\mathcal{O}_{K_5}$

The clutter $\mathcal{C}_{2,3}$ is minimally nonideal. Clutter $\mathcal{C}_{2,3}$ is the (non-binary) clutter on ground set $\{1, 2, 3\}$ with elements $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. The blocker of $\mathcal{C}_{2,3}$ is itself; $b(\mathcal{C}_{2,3}) = \mathcal{C}_{2,3}$. The minimum cardinality elements of $\mathcal{C}_{2,3}$ are just $\mathcal{C}_{2,3}$, likewise for the blocker. Thus $M(\overline{\mathcal{C}_{2,3}})[M(\overline{b(\mathcal{C}_{2,3})})]^T = J + I$. One way to interpret this matrix equation is to note that $F \in \mathcal{C}_{2,3}$ has intersection 2 with exactly one element in $b(\mathcal{C}_{2,3})$ (the element corresponding to itself) and intersection 1 with the other two elements of $b(\mathcal{C}_{2,3})$.

The clutter $\mathcal{O}_{K_5}$ is minimally nonideal. The minimum cardinality elements of $\mathcal{O}_{K_5}$ are the triangles of K_5. The minimum cardinality elements of $b(\mathcal{O}_{K_5})$ are the union of a triangle of K_5 together with the non-adjacent edge. Thus $M(\overline{\mathcal{O}_{K_5}})[M(\overline{b(\mathcal{O}_{K_5})})]^T = J + 2I$. One way to interpret this matrix equation is to note that each triangle T of K_5 has intersection 3 with the element of the blocker consisting of T and the opposite edge and intersection 1 with every other minimum cardinality blocker element.

3.4.2   Useful Corollaries

Given clutter $\mathcal{F}$ we denote by $\bar{\mathcal{F}}$ the clutter formed by the sets of minimum cardinality in $\mathcal{F}$. Let $(\mathcal{F}, \mathcal{K})$ be a blocking pair that is Lehman. We say that $F \in \bar{\mathcal{F}}$ and $K \in \check{\mathcal{K}}$ are *mates* if $|F \cap K| \geq 2$. Equation (3.1) implies every $F \in \bar{\mathcal{F}}$ has a unique mate K and $|F \cap K| = r\ell - n + 1 := q$. Moreover, if $F \in \bar{\mathcal{F}}$ and $K' \in \bar{\mathcal{K}}$ are not mates then $|F \cap K'| = 1$. Since $\bar{\mathcal{F}}$ is binary, elements of $\bar{\mathcal{F}}$ and $\bar{\mathcal{K}}$ intersect with odd parity and thus $|F \cap K| \geq 3$.

Interpreting equation (3.1) combinatorially leads to many corollaries of Lehman's Theorem. First note that $M(\bar{\mathcal{F}})$ and $[M(\bar{\mathcal{K}})]^T$ commute (Theorem 3.3.7) and $(J+dI)^T = J+dI$; thus $M[\bar{\mathcal{F}}]^T M[\bar{\mathcal{K}}] = J + dI$.

Corollary 3.4.1. *Let $e \in E(\mathcal{F})$ be a column index of $M[\bar{\mathcal{K}}]$. Let $F_1, \dots F_\ell$ be the characteristic sets of the rows of $M[\bar{\mathcal{F}}]$ whose indices are given by the characteristic set of column e of $M[\bar{\mathcal{K}}]$. Then $F_1 \dots F_\ell$ pairwise intersect in at most $\{e\}$; moreover, exactly q of $F_1 \dots F_\ell$ contain $\{e\}$. Lastly, $E = F_1 \cup \dots \cup F_\ell$.*

Proof. Since $M[\bar{\mathcal{F}}]^T M[\bar{\mathcal{K}}]$, for each column index j of $M[\bar{\mathcal{K}}]$, we have $M[\bar{\mathcal{F}}]^T \mathrm{col}(M[\bar{\mathcal{K}}], j) = \mathbb{1} + de_j$ where e_j is the jth standard basis vector and $\mathrm{col}(M[\bar{\mathcal{K}}], j)$ denotes the jth column of $M[\bar{\mathcal{K}}]$. The corollary follows. $\square$

Corollary 3.4.2. *Let $e \in E(\mathcal{F})$ be a column index of $M[\bar{\mathcal{F}}]$. Let $K_1, \dots K_r$ be the characteristic sets of the rows of $M[\bar{\mathcal{K}}]$ whose indices are given by the characteristic set of column e of $M[\bar{\mathcal{F}}]$. Then $K_1 \dots K_r$ pairwise intersect in at most $\{e\}$; moreover, exactly q of $K_1 \dots K_r$ contain $\{e\}$. Lastly, $E = K_1 \cup \dots \cup K_r$.*

Proof. Similar to proof of Corollary 3.4.1. $\square$

Corollary 3.4.3. *For each $e \in E$ there are precisely q pairs $\{(F_1, K_1), \dots (F_q, K_q)\}$ such that $e \in F_i \cap K_i$. For each distinct $e, f \in E$, there is exactly one pair (F, K) with $e \in F$, $f \in K$.*

Proof. This proof appears in [28]. Let $e \in E$. Since $(M(\bar{\mathcal{F}}))[M(\bar{\mathcal{K}})]^T$, column e of $M(\bar{\mathcal{K}})$ must have exactly q 1s in rows corresponding to the characteristic set of column e of $M(\bar{\mathcal{F}})$. Moreover, column f must have exactly one 1 in the rows corresponding to the characteristc set of column e of $M(\bar{\mathcal{F}})$. $\square$

Corollary 3.4.4. *For fixed $e \in E$, if $\{(F_1, K_1), \dots (F_q, K_q)\}$ are the mates such that $e \in F_i \cap K_i$ then $\{F_1, \dots F_q, K_1 \dots K_q\}$ pairwise intersect in at most $\{e\}$ except for $|F_i \cap K_i| = q$.*

Proof. Now $F_i \cap B_j = \{e\}$ since $|C_i \cap B_j| = 1$ for all distinct $i, j \in \{1 \ldots q\} \subseteq \{1 \ldots n\}$. Similarly, the facts $|F_i \cap B_i| = q$ and $e \in F_i \cap B_i$ let us conclude that $|(F_i \cap K_i)| = q$. It remains to show that $C_i \cap C_j = \{e\}$ for all distinct $i, j \in \{1, \ldots q\}$ and $B_i \cap B_j = \{e\}$ for all distinct $i, j \in \{1, \ldots q\}$. Suppose there exists $f \in E - e$ such that $f \in C_i \cap C_j$. Then $e \in B_i$ and $f \in C_i$, and $e \in B_j$ and $f \in C_j$. This contradicts the second conclusion of Corollary 3.4.3. Therefore such f cannot exist. Thus $C_i \cap C_j = \{e\}$ for all distinct $i, j \in \{1 \ldots q\}$. The analysis is analogous for showing $B_i \cap B_j = \{e\}$ for distinct $i, j \in \{1 \ldots q\}$. $\square$

Corollary 3.4.5. *There exists no set K such that $|K \cap F| = 1$ for all $F \in \bar{\mathcal{F}}$.*

Proof. By way of contradiction, suppose there exists K such that $|K \cap F| = 1$ for all $F \in \bar{\mathcal{F}}$. By Corollary 3.4.1, for $e \in K$, there exists a collection of sets $\{F_1 \ldots F_\ell\}$ such that $E = F_1 \cup \ldots \cup F_\ell$ and $e \in F_1 \cap F_2$. This implies $|K| \leq \ell - 1$. Again by Corollary 3.4.1, for $e' \notin K$, there exists a collection of sets $\{K_1' \ldots K_\ell'\}$ that pairwise intersect in at most $\{e\}$. This implies $|K| \geq \ell$, a contradiction. Conclude that no such K can exist. $\square$

The following results appear in [18] and apply only to binary clutters. The statements were given for minimally nonideal clutters, but the proof works for the more general class of Lehman clutters.

Corollary 3.4.6. *Let $(\mathcal{F}, \mathcal{K})$ be a blocking pair that is binary and Lehman, and let $e \in E(\mathcal{F})$. Then there exists $F_1, F_2, F_3 \in \bar{\mathcal{F}}$ that pairwise intersect exactly in e. Moreover, the mates K_1, K_2, K_3 of F_1, F_2, F_3 pairwise intersect exactly in e.*

Proof. This proof appears in [19]. Since $q \geq 3$ because $\mathcal{F}$ is binary, result follows from Corollary 3.4.4. $\square$

Corollary 3.4.7. *Let $(\mathcal{F}, \mathcal{K})$ be a blocking pair that is binary and Lehman.*

1. *For all $F_1, F_2 \in \bar{\mathcal{F}}$ and $F \in \mathcal{F}$ where $F \subseteq F_1 \cup F_2$ either $F = F_1$ or $F = F_2$.*
2. *For all $K_1, K_2 \in \bar{\mathcal{K}}$ and $K \in \mathcal{K}$ where $K \subseteq K_1 \cup K_2$ either $K = K_1$ or $K = K_2$.*

Proof. Since $\mathcal{F}$ is binary, there exists $F' \subseteq F_1 \triangle F_2 \triangle F$ such that $F' \in \mathcal{F}$. If $e \in F \cap F'$ and $F_1 \triangle F_2 \triangle F$, then $e \notin F_1 \triangle F_2$. Thus $F \cup F' \subseteq F_1 \cup F_2$ and $F \cap F' \subseteq F_1 \cap F_2$. It follows that $|F| + |F'| \leq |F_1| + |F_2|$ and thus $F, F' \in \mathcal{F}$. Let B be the mate of F. Since $\mathcal{F}$ is binary, $|B \cap F|$ must be odd and hence it is at least 3. Then either $|F_1 \cap B| \geq 2$ or $|F_2 \cap B| \geq 2$ and since B has a unique mate in $\bar{\mathcal{F}}$, it must be that $F = F_1$ or $F = F_2$. Similarly, one can show that if $F = F_1$, then $F' = F_2$ and if $F = F_2$, then $F' = F_1$. The proof is the same for $K_1, K_2, K \in \bar{\mathcal{K}}$. $\square$

In the remainder of this thesis, an above corollary will be referenced by "Lehman Corollary" followed by the corollary number.

Chapter 4

Applications to Odd-st-Walks and st-T-Cuts

4.1   Chapter Overview

This chapter focuses on two cases: (1) $\mathcal{F}$ is a clutter of odd-st-walks, and, (2) $\mathcal{F}$ is a clutter of st-T-cuts.

Given signed graph (G, Σ), $s, t \in V(G)$, and weight function $w \in \mathbb{Z}_+^{E(G)}$, let $\mathcal{F}$ be the clutter of odd-st-walks of (G, Σ). Whenever $\mathcal{F}$ has no $\mathcal{O}_{K_5}$ or $\mathcal{F}_7$ minor, there exists a fractional w-packing and integer w-cover of $\mathcal{F}$ with the same value (Theorem 2.4.1). Using the constructive version of Lehman's Theorem, one can either find a fractional w-packing and integer w-cover of the same value or show that $\mathcal{O}_{K_5}$ or $\mathcal{L}_7$ is a minor of $\mathcal{F}$. Moreover, this can be done in polynomial time, proving below Theorem 4.1.1 (which is equivalent to Theorem 1.3.2).

Theorem 4.1.1. *Let (G, Σ) be a signed graph and $w \in \mathbb{Z}_+^{E(G)}$. Let $s, t \in V(G)$ and $\mathcal{F}$ be the clutter of odd-st-walks of (G, Σ). Then in time polynomial in $|V(G)| + \langle w \rangle$, one can either find*

1. *$I, J \subseteq E(\mathcal{F})$ such that $\mathcal{F}/I \setminus J$ is isomorphic to $\mathcal{O}_{K_5}$ or $\mathcal{L}_7$, or*
2. *an integer solution to (2.5) and a solution to (2.6) with the same value.*

Given graft (G, T), $s, t \in V(G)$, and weight function $w \in \mathbb{Z}_+^{E(G)}$, let $\mathcal{F}$ be the clutter of st-T-cuts of (G, T). Whenever $\mathcal{F}$ has no $b(\mathcal{O}_{K_5})$ or $\mathcal{L}_7$ minor, there exists a fractional

w-packing and integer w-cover of $\mathcal{F}$ with the same value (Theorem 2.4.4). Using the constructive version of Lehman's Theorem, one can either find a fractional w-packing and integer w-cover of the same value or show that $b(\mathcal{O}_{K_5})$ of $\mathcal{L}_7$ is a minor of $\mathcal{F}$. Moreover, this can be done in polynomial time, proving below Theorem 4.1.2 (which is equivalent to Theorem 1.3.3).

Theorem 4.1.2. *Let (G,T) be a graft and $w \in \mathbb{Z}_+^{E(G)}$. Let $s,t \in V(G)$ and $\mathcal{F}$ be the clutter of st-T-cuts in (G,T). Then in time polynomial in $|V(G)| + \langle w \rangle$, one can either find*

1. *$I, J \subseteq E(\mathcal{F})$ such that $\mathcal{F}/I \setminus J$ is isomorphic to $b(\mathcal{O}_{K_5})$ or $\mathcal{L}_7$, or*
2. *an integer solution to (2.5) and a solution to (2.6) with the same value.*

This chapter is dedicated to proving Theorems 4.1.1 and 4.1.2. Section 4.2 introduces signed graph and graft minor operations and shows them correspond to minor operations for clutters of odd-st-walks and clutters of st-T-cuts respectively. Section 4.3 provides a proof outline; the remaining sections include the more technical details.

4.2 Minors in Signed Graphs and Grafts

Let (G,Σ) be a signed graph with $s,t \in V(G)$. Although minor operations in signed graphs have already been defined, here we must take care to retain s and t. Updating Σ to $\Sigma \triangle \delta(S)$ for $s,t \in S \subseteq V$ preserves the set of odd st-walks in (G,Σ). We call this operation *resigning* and refer to any Σ' obtained in this way as a *signature* of (G,Σ). Since (G,Σ) and (G,Σ') have the same set of odd st-walks, we will view these two signed graphs as equivalent. Accordingly, we do not account explicitly for the resigning operation. Given (G,Σ) and $e \in E$, we define $(G,\Sigma) \setminus e$ as $(G \setminus e, \Sigma - e)$ and call this operation *deletion*. Given (G,Σ) and $e \in E$ that is not an odd loop, suppose we resign to obtain Σ' such that $e \notin \Sigma'$. We define $(G,\Sigma)/e$ as $(G/e, \Sigma')$ and call this operation *contraction*. If contracting edge $e = sx$, call the resulting vertex s. If contracting edge $e = tx$, call the resulting vertex t. If contracting edge $e = st$, call the resulting vertex $s = t$. Any (G',Σ') obtained by applying a sequence of deletions and contractions is called a *minor* of (G,Σ).

We make some notes about this definition. Minor operations can be applied in any order and so we unambiguously denote by $(G,\Sigma)/I \setminus J$ the signed graph obtained by contracting all edges in $I \subseteq E$ and deleting all edges in $J \subseteq E$ (and resigning in whatever way avoids contracting signature edges). Necessarily I does not contain an odd circuit. Note that if $s = t$, the minor operations defined here correspond exactly to the signed graph

minor operations defined in Section 1.1.3. The following proposition shows performing contractions and deletions in a signed graph corresponds to performing the same operations in the underlying clutter of odd-st-walks.

Proposition 4.2.1. *If $\mathcal{F}$ is the clutter of odd st-walks of (G, Σ) then $\mathcal{F}/I \setminus J$ is the clutter of odd st-walks of $(G, \Sigma)/I \setminus J$.*

Proof. It suffices to show that $\mathcal{F} \setminus e$ is the clutter of odd-st-walks of $(G, \Sigma) \setminus e$ and $\mathcal{F}/e$ is the clutter of odd-st-walks of $(G, \Sigma)/e$.

The clutter $\mathcal{F} \setminus e$ is defined as $\{F \in \mathcal{F} : e \notin F\}$. In other words $\mathcal{F} \setminus e$ contains exactly the odd-st-walks of (G, Σ) that do not use edge e. Deleting edge e from (G, Σ) results in removing all odd-st-walks that use e and preserving all other odd-st-walks. Therefore, $\mathcal{F} \setminus e$ is indeed the clutter of odd-st-walks of $(G, \Sigma) \setminus e$.

Without loss of generality, assume $e \notin \Sigma$ since otherwise we can resign and preserve the set of odd-st-walks. The clutter $\mathcal{F}/e$ is defined as the minimal sets of $\{F - e : F \in \mathcal{F}\}$. In other words, sets of $\mathcal{F}/e$ come in two sorts: odd st-walks of (G, Σ) that do not use e and odd-st-walks of (G, Σ) using e with the edge e removed. Since $e \notin \Sigma$, the parity of $D \subseteq E$ is the same in both (G, Σ) and $(G, \Sigma)/e$. Thus, contracting e in (G, Σ) preserves the odd-st-walks that do not use e. Since $e \notin \Sigma$, if F is an odd-st-walk using e, then $F - e$ is an odd-st-walk of $(G, \Sigma)/e$. Therefore, $\mathcal{F}/e$ is indeed the clutter of odd-st-walks of $(G, \Sigma)/e$. □

Minor operations can also be defined for grafts. Let (G, T) be a graft with $s, t \in V(G)$. Suppose $e \in E(G) \setminus \{st\}$ is not an odd bridge (an edge whose removal breaks (G, T) into two components each of which contains an odd number of T vertices). We define $(G, T) \setminus e$ as $(G \setminus e, T)$ and call this operation *deletion*. For $e = xy \in E(G)$, let $G' = G/e$. If z is the vertex obtained by contracting e, then let $T' = T - \{x, y\} \cup \{z\}$ if exactly one of x, y is in T and otherwise let $T' = T - \{x, y\}$. We define $(G, T)/e$ as (G', T') and call this operation *contraction*. If contracting edge $e = sx$, call the resulting vertex s. If contracting edge $e = tx$, call the resulting vertex t. Any (G', T') obtained by applying a sequence of deletions and contractions is called a *minor* of (G, T).

Again, minor operations can be applied in any order and so we can unambiguously denote by $(G, T)/I \setminus J$ the graft obtained by contracting all edges in $I \subseteq E$ and deleting all edges in $J \subseteq E$. Necessarily $G \setminus I$ does not contain a T-odd component. The following proposition shows contraction in the graft corresponds to deletion in the underlying clutter of st-T-cuts and deletion in the grafts corresponds to contraction in the underlying clutter of st-T-cuts.

Proposition 4.2.2. *If $\mathcal{F}$ is the clutter of st-T-cuts of (G,T) then $\mathcal{F}/I \setminus J$ is the clutter of st-T-cuts of $(G,T) \setminus I/J$.*

Proof. It suffices to show that $\mathcal{F} \setminus e$ is the clutter of st-T-cuts of $(G,T)/e$ and $\mathcal{F}/e$ is the clutter of st-T-cuts of $(G,\Sigma) \setminus e$.

The clutter $\mathcal{F} \setminus e$ is defined as $\{F \in \mathcal{F} : e \notin F\}$. In other words, $\mathcal{F} \setminus e$ contains exactly the st-T-cuts of (G,T) that do not use e. Suppose $\delta(S)$ for $S \subseteq V$ is an st-T-cut of (G,T) and let $(G',T') = (G,T)/e$. If $e \in \delta(S)$, there is no corresponding st-T'-cut of (G',T') since $\delta(S) \setminus e$ is not a cut of (G',T'). If $e = xy \notin \delta(S)$, then $\delta(S)$ is a st-T'-cut of (G',T'). Moreover, either $x,y \in S$ or $x,y \notin S$. In both cases, $|S \cap T'|$ is odd since contracting $\{x,y\}$ to $\{z\}$ is defined so that the parity of $T \cap \{x,y\}$ and the parity of $T' \cap \{z\}$ are the same. Thus $\delta(S)$ is an st-T'-cut of (G',T').

The clutter $\mathcal{F}/e$ is defined as the minimal elements of $\{F - e : F \in \mathcal{F}\}$. In other words, sets of $\mathcal{F}/e$ come in two sorts: st-T-cuts of (G,T) that do not use e and st-T-cuts of (G,T) using e with the edge e removed. If $\delta(S)$ is an st-T-cut of (G,T) that does not use e, it is clear the $\delta(S)$ is an st-T-cut of $(G,T) \setminus e$ that does not use e. Similarly, if $\delta(S)$ is an st-T-cut of (G,T) using e, $\delta(S) \setminus e$ is an st-T-cut of $(G,T) \setminus e$. $\qquad\square$

It follows that we can consider signed graph and graft minor operations instead of clutter operations whenever it is convenient.

4.3   Proof Outline

The proofs for Theorem 4.1.1 (regarding clutters of odd-st-walks) and Theorem 4.1.2 (regarding clutters of st-T-cuts) follow the same outline. In the interest of clarity and at the cost of repetition, full outlines of both proofs follow.

4.3.1   Proof Outline: Theorem 4.1.1

Recall that the separation problem for P and a point $\bar{x} \in \mathbb{Q}^n$ is to either determine that $\bar{x} \in P$, or to find a separating constraint $a^T x \leq \beta$ (i.e., $a^T \bar{x} > \beta$ and $a^T x \leq \beta$ for all $x \in P$). In order for the constructive version of Lehman's Theorem (Theorem 3.1.3) to give a polynomial time algorithm, the separation problem for $Q(\mathcal{F})$ must be solvable in polynomial time. For odd-st-walk covering polytopes, polynomial time separation follows almost immediately from being able to find minimum length circuits and paths in graphs

subject to a parity condition (see [16]). Section 4.4 describes the separation algorithm and shows correctness; the following proposition appears.

Proposition 4.3.1. *Let (G, Σ) be a signed graph and let $s, t \in V(G)$. Let $\mathcal{F}$ be the clutter of odd st-walks of (G, Σ). Given $\bar{x} \in \mathbb{Q}^{E(G)}$, we can solve the separation problem for $\bar{x} \in \mathbb{Q}^n$ and $Q(\mathcal{F})$ in time polynomial in $|V(G)| + \langle \bar{x} \rangle$.*

The following proposition is obtained by combining (6.5.9) and (6.5.15) in [16] and allows us to find not only optimal solutions to the packing and covering LPs, but also defining constraints.

Proposition 4.3.2. *Let $\mathcal{F}$ be a clutter and suppose that $Q(\mathcal{F}) \subseteq \mathbb{R}^n$ is given by a separation oracle. If $w \in \mathbb{Z}_+^{E(\mathcal{F})}$, then in oracle polynomial time in $n + \langle w \rangle$ we can find an optimal solution $\bar{y}$ to (2.6) and an extreme point $\bar{x}$ of $Q(\mathcal{F})$ that is optimal for (2.5) together with a set of n constraints of $Q(\mathcal{F})$ defining $\bar{x}$.*

Let $\mathcal{F}$ be the clutter of odd st-walks of (G, Σ). Since the separation problem for $\mathcal{F}$ is polynomially solvable by Proposition 4.3.1, it follows immediately from Proposition 4.3.2 that we can find in time polynomial in $|V(G)| + \langle w \rangle$

1. an optimal solution $\bar{y}$ to (2.6) (the odd-st-walk packing LP),
2. an extreme point $\bar{x}$ of $Q(\mathcal{F}) \equiv$ (2.5) (the odd-st-walk covering LP), and,
3. a set of n constraints of $Q(\mathcal{F})$ defining $\bar{x}$.

If $\bar{x}$ is integral, we have achieved the second outcome of Theorem 4.1.1; $\bar{x}$ and $\bar{y}$ are an integer w-cover and fractional w-packing of odd-st-walks of (G, Σ) with the same value. Otherwise, we aim to find $I, J \subseteq E(\mathcal{F})$ such that $\mathcal{F}/I \setminus J$ is isomorphic to $\mathcal{O}_{K_5}$ or $\mathcal{L}_7$. Proposition 4.3.1 gives a polynomial time separation oracle for $Q(\mathcal{F})$, $\bar{x}$ is a fractional extreme point of $Q(\mathcal{F})$, and we know n facets of $Q(\mathcal{F})$ that define $\bar{x}$. Hence the constructive version of Lehman's Theorem (Theorem 3.1.3) applies; in polynomial time we can find $I, J \subseteq E(\mathcal{F})$ such that $\mathcal{F}' = \mathcal{F}/I \setminus J$ is a Lehman clutter. The first outcome ($\mathcal{F}' = \mathcal{J}_s$ for $s \geq 2$) cannot occur since the clutter of odd-st-walks is binary but $\mathcal{J}_s$ is not. Theorem 3.1.3 also lets us find the minimum cardinality sets of $\mathcal{F}'$. By Proposition 4.2.1, $\mathcal{F}'$ is the clutter of odd-st-walks of $(G', \Sigma') = (G, \Sigma)/I \setminus J$.

Given Lehman clutter $\mathcal{F}'$, Lemma 4.3.3 below proves we can efficiently show $\mathcal{O}_{K_5}$ or $\mathcal{L}_7$ is a minor of $\mathcal{F}'$. The proof is a constructive variant of known structures (see [28] and [19]). A proof of Lemma 4.3.3 appears in Section 4.5.

Lemma 4.3.3. *Let (G, Σ) be a signed graph and let $s, t \in V(G)$. Let $\mathcal{F}$ be the clutter of odd st-walks of (G, Σ). Suppose that $\mathcal{F}$ is a Lehman clutter and that we are given the minimum*

cardinality sets of $\mathcal{F}$. Then in time polynomial in $|V(G)|$ we can find $I, J \subseteq E(\mathcal{F})$ such that $\mathcal{F}/I \setminus J$ is isomorphic to $\mathcal{O}_{K_5}$ or $\mathcal{L}_7$.

4.3.2 Proof Outline: Theorem 4.1.2

Polynomial time separation for st-T-cut covering polytopes was known previously (see [16]). Section 4.4 describes the separation algorithm and shows correctness; the following proposition appears.

Proposition 4.3.4. *Let (G, T) be a graft and let $s, t \in V(G)$. Let $\mathcal{F}$ be the clutter of st-T-cuts of (G, Σ). Given any $\bar{x} \in \mathbb{Q}^{E(G)}$, we can solve the separation problem for $\bar{x} \in \mathbb{Q}^n$ and $Q(\mathcal{F})$ in time polynomial in $|V(G)| + \langle \bar{x} \rangle$.*

Let $\mathcal{F}$ be the clutter of st-T-cuts of (G, T). Since the separation problem for $\mathcal{F}$ is polynomially solvable by Proposition 4.3.4, it follows immediately from Proposition 4.3.2 we can find in time polynomial in $|V(G)| + \langle w \rangle$

1. an optimal solution $\bar{y}$ to (2.6) (the st-T-cut packing LP),
2. an extreme point $\bar{y}$ of $Q(\mathcal{F}) \equiv$ (2.5) (the st-T-cut covering LP), and,
3. a set of n constraints of $Q(\mathcal{F})$ defining $\bar{x}$.

If $\bar{x}$ is integral, we have achieved the second outcome of Theorem 4.1.2; $\bar{x}$ and $\bar{y}$ are an integer w-cover and fractional w-packing of st-T-cuts of (G, T) with the same value. Otherwise, we aim to find $I, J \subseteq E(\mathcal{F})$ such that $\mathcal{F}/I \setminus J$ is isomorphic to $b(\mathcal{O}_{K_5})$ or $\mathcal{L}_7$. Proposition 4.3.4 gives a polynomial time separation oracle for $Q(\mathcal{F})$, $\bar{x}$ is a fractional extreme point of $Q(\mathcal{F})$, and we know n facets of $Q(\mathcal{F})$ that define $\bar{x}$. Hence the constructive version of Lehman's Theorem (Theorem 3.1.3) applies; in polynomial time we can find $I, J \subseteq E(\mathcal{F})$ such that $\mathcal{F}' = \mathcal{F}/I \setminus J$ is a Lehman clutter. The first outcome ($\mathcal{F}' = \mathcal{J}_s$ for $s \geq 2$) cannot occur since the clutter of st-T-cuts is binary but $\mathcal{J}_s$ is not. Theorem 3.1.3 also lets us find the minimum cardinality sets of $\mathcal{F}'$. By Proposition 4.2.2, $\mathcal{F}'$ is the clutter of st-T-cuts of $(G', T') = (G, T)/J \setminus I$.

Given Lehman clutter $\mathcal{F}'$, Lemma 4.3.5 below proves we can efficiently show $b(\mathcal{O}_{K_5})$ or $\mathcal{L}_7$ is a minor of $\mathcal{F}'$. The proof is a constructive variant of known structures (see [19]). A proof of Lemma 4.3.5 appears in Section 4.6.

Lemma 4.3.5. *Let (G, T) be a graft and let $s, t \in V(G)$. Let $\mathcal{F}$ be the clutter of st-T-cuts of G. Suppose that $\mathcal{F}$ is a Lehman clutter and that we are given the minimum cardinality sets of $\mathcal{F}$. Then in time polynomial in $|V(G)|$ we can find $I, J \subseteq E(\mathcal{F})$ such that $\mathcal{F}/I \setminus J$ is isomorphic to $b(\mathcal{O}_{K_5})$ or $\mathcal{L}_7$.*

4.4 Solving the Separation Problems

4.4.1 The Separation Problem for Odd-st-Walks

Let (G, Σ) be a signed graph, $s, t \in V(G)$, and $\mathcal{F}$ be the clutter of odd st-walks of (G, Σ). Given $\bar{x} \in \mathbb{Q}^n$, we require a polynomial time method to show $\bar{x} \in Q(\mathcal{F})$ or find a separating hyperplane.

It is straightforward to check that $0 \leq \bar{x}_e \leq 1$ for all $e \in E$ in polynomial time. We also need to verify $\sum(\bar{x}_e : e \in F) \geq 1$ for all $F \in \mathcal{F}$. Since $\mathcal{F}$ may have exponentially many members, we can not verify all of these constraints in the straightforward way. Suppose $F^* \in \mathcal{F}$ is a minimum weight odd-st-walk with respect to weight function $\bar{x}$. If $\sum(\bar{x}_e : e \in F^*) < 1$, then letting a be the characteristic vector of F^* and b be 1, $a^T x \geq b$ is a separating hyperplane for $\bar{x}$ and $Q(\mathcal{F})$. If $\sum(\bar{x}_e : e \in F^*) \geq 1$ then for all $F \in \mathcal{F}$

$$\sum(\bar{x}_e : e \in F) \geq \sum(\bar{x}_e : e \in F^*) \geq 1,$$

where the first inequality holds because F^* is a minimum weight member of $\mathcal{F}$. Therefore, we can conclude $\bar{x} \in Q(\mathcal{F})$. It remains to show that we can find a minimum weight odd st-walk in polynomial time.

This problem is closely related to finding a minimum weight circuit or path in a graph subject to a parity condition. In graph $G = (V, E)$, the parity of $E' \subseteq E$ is simply $|E'|$, while in a signed graph (G, Σ), the parity of $E' \subseteq E$ is $|E' \cap \Sigma|$. However, finding an odd circuit in (G, Σ) is reducible to finding an odd circuit in G' where G' is obtained from G by replacing every edge $e \notin \Sigma$ by a path of length 2. Therefore, a polynomial time algorithm for finding minimum weight circuits and paths in graphs subject to a parity condition implies a polynomial time algorithm for finding minimum weight circuits and paths in signed graphs subject to a parity condition. Algorithms for these problems in graphs are known [16]. For completeness, a precise statement and proof follow.

Theorem 4.4.1. *Given a graph $G = (V, E)$, $s, t \in V(G)$, and weight function $x \in \mathbb{Q}_+^E$, the following problems are solvable in polynomial time.*

1. *Find a minimum weight odd circuit through fixed $s \in V$.*
2. *Find a minimum weight odd circuit in G.*
3. *Find a minimum weight even circuit in G.*
4. *Find a minimum weight even st-path in G.*
5. *Find a minimum weight odd st-path in G.*

Proof. Let the auxiliary graph G' be obtained from G by splitting each vertex $v \in V$ into vertices v_1 and v_2. If edge $e = xy \in E(G)$, then add edges x_1y_2 and x_2y_1 to $E(G')$, giving both weight w_e. Note that G' can be computed in polynomial time.

The shortest odd circuit through fixed s in G is the shortest s_1, s_2 path in G'. Since this shortest path can be found in polynomial time (for example, by Dijkstra's algorithm), finding a minimum weight odd circuit through s can be done in polynomial time.

Apply the algorithm for finding a minimum weight circuit C_s through fixed $s \in V$ to each vertex to find a minimum weight odd circuit in G by simply returning $C = \text{argmin}\{C_v : v \in V\}$. Since this requires n applications of a polynomial time algorithm, this too runs in polynomial time.

The minimum weight even circuit in G must use some edge $e \in E(G)$. Note that if $\tilde{C}$ is an even circuit, then $\tilde{C} \setminus e$ is an odd path. Thus, the minimum weight even circuit problem can be solved by finding for each $e = xy \in E$ the shortest odd xy-path.

Auxiliary graph G' can be used to find the shortest odd and even paths in G. To find a minimum weight even path from s to t, find a shortest path from s_1 to t_1. To find a minimum weight odd path from s to t, find a shortest path from s_1 to t_2. Thus, a minimum weight even st-path, minimum weight odd st-path and so also a minimum weight even cycle can be found in polynomial time. $\qquad\square$

Finding a minimum weight odd-st-walk in polynomial time follows almost immediately from the preceding theorem.

Claim 4.4.2. *Given (G, Σ) such that $s, t \in V(G)$ and weight function $x \in \mathbb{Q}_+^E$, there exists a polynomial time algorithm to find a minimum weight odd-st-walk of (G, Σ).*

Proof. Recall that an odd st-walk is either an odd st-path or the edge disjoint union of an even st-path P and an odd circuit C such that $V(P)$ and $V(C)$ have at most one vertex in common. Suppose P_{odd} is a minimum weight odd st-path in (G, Σ), P_{even} is a minimum weight even st-path in (G, Σ), and C_{odd} is a minimum weight odd circuit in (G, Σ).

If $w(P_{odd}) \leq w(P_{even} \cup C_{odd})$ then P_{odd} is the minimum odd st-walk since any odd st-walk that is the union of an even path and odd circuit must have weight at least $w(P_{even}) + w(C_{odd})$.

Otherwise $w(P_{even} \cup C_{odd}) \leq w(P_{odd})$. If $P_{even} \cup C_{odd}$ is the edge disjoint union of an odd circuit and even st-path sharing at most one vertex, then $P_{even} \cup C$ is an odd st-walk and thus the minimum odd st-walk. Otherwise, $P_{even} \cup C_{odd}$ is not an odd st-walk. Then P_{even}

and C_{odd} must share at least 2 vertices and so $P_{even} \cup C_{odd}$ properly contains some odd st-path P'. Since $w(P') < w(P_{even} \cup C_{odd})$, the minimum odd st-walk must be an odd st-path, a contradiction. Thus, we can find a minimum weight odd st-walk in polynomial time by finding P_{odd}, P_{even} and C_{odd} (in polynomial time as per Theorem 4.4.1) and choosing P_{odd} if $w(P_{odd}) \leq w(P_{even} \cup C_{odd})$ and choosing $P_{even} \cup C_{odd}$ otherwise. $\qquad\square$

4.4.2 The Separation Problem for st-T-Cuts

Analogously to the analysis for odd-st-walks, it suffices to show that we can find a minimum weight st-T-cut in polynomial time. An early proof of this result is found in [16] and follows from a straightforward uncrossing argument; this argument is, in fact, known to work for a more general problem.

Let $G = (V, E)$ be a graph and let $T_1, T_2 \subseteq V$ be such that $|T_1|$ and $|T_2|$ are even. Let $w \in \mathbb{R}_+^E$ be a weight function on the edges of G. A T_1, T_2-*cut* is a set of edges of the form $\delta(U)$ where $|U \cap T_1|$ and $|U \cap T_2|$ are odd. Note that choosing $T_1 = T$ and $T_2 = \{s, t\}$ in the below theorem implies we can find a minimum weight st-T-cut in polynomial time.

Claim 4.4.3. *Given graph $G = (V, E)$, $T_1, T_2 \subseteq V$ with $|T_1|, |T_2|$ even and $w \in \mathbb{R}_+^E$, there exists a polynomial time algorithm to find a minimum weight T_1, T_2-cut.*

Proof. Recall that a T-cut is $\delta(U)$ such that $|U \cap T|$ is odd. It is well known that finding a minimum weight T-cut can be done in polynomial time [16].

First, find a minimum weight T_1-cut $\delta(U)$. If $|U \cap T_2|$ is odd, then $\delta(U)$ is a minimum T_1, T_2-cut. Otherwise, we require the following uncrossing property: "There exists a minimum weight T_1, T_2-cut $\delta(X)$ of G such that $X \subseteq U$ or $X \subseteq \bar{U}$.".

Proof of uncrossing property:

Let X be a minimum weight T_1, T_2-cut. If the uncrossing property does not hold, then U and X *cross*. This means $U \setminus X \neq \emptyset$ and $X \setminus U \neq \emptyset$.

Since $|X \cap T_1|$ is odd, $|X \cap U \cap T_1|$ and $|X \cap \bar{U} \cap T_1|$ have distinct parities. Since $|\bar{U} \cap T_1|$ is odd, $|X \cap \bar{U} \cap T_1|$ and $|\bar{X} \cap \bar{U} \cap T_1|$ have distinct parities. Thus $|X \cap U \cap T_1|$ and $|\bar{X} \cap \bar{U} \cap T_1|$ have the same parity. If they are both of odd parity, continue.

Otherwise, since $|U \cap T_1|$ is odd, $|X \cap U \cap T_1|$ and $|\bar{X} \cap U \cap T_1|$ have distinct parities. It follows that $|\bar{X} \cap U \cap T_1|$ and $|X \cap \bar{U} \cap T_1|$ are both of odd parity. If this is the case, since $\delta(X)$ and $\delta(\bar{X})$ are minimum weight T_1, T_2-cuts, we can exchange freely the roles of X and $\bar{X}$ and thus assume that $|X \cap U \cap T_1|$ and $|\bar{X} \cap \bar{U} \cap T_1|$ both have odd parity.

Since $|X \cap T_2|$ is odd, $|X \cap U \cap T_2|$ and $|X \cap \bar{U} \cap T_2|$ have distinct parities. Since $|\bar{U} \cap T_2|$ is even, $|X \cap \bar{U} \cap T_2|$ and $|\bar{X} \cap \bar{U} \cap T_2|$ have the same parity. Thus we may assume (after possibly interchanging both the role of U and $\bar{U}$ and X and $\bar{X}$) that $|X \cap U \cap T_2|$ is odd and $|\bar{X} \cap \bar{U} \cap T_2|$ is even. See Figure 4.1 for a representative figure of the necessary parities.
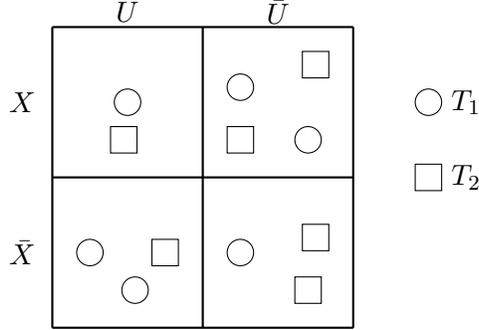
Figure 4.1: Uncrossing Illustration: $|X \cap U \cap T_1|$ is odd, etc.

It is well-known the weight function is submodular and thus

$$w(\delta(X \cap U)) + w(\delta(\bar{X} \cap \bar{U})) \leq w(\delta(U)) + w(\delta(X)). \tag{4.1}$$

Since $\delta(U)$ is a minimum T_1-cut and $\bar{X} \cap \bar{U}$ is also a T_1-cut, $w(\delta(U)) \leq w(\delta(\bar{X} \cap \bar{U}))$. Since $\delta(X)$ is a minimum T_1, T_2-cut and $X \cap U$ is also a T_1, T_2-cut, $w(\delta(X)) \leq w(\delta(X \cap U))$. Thus equality must hold in (4.1). In particular $w(\delta(X)) = w(\delta(X \cap U))$ and thus $X \cap U$ is a minimum T_1, T_2-cut of G that is contained in U. $\square$

Returning to the main statement, if U is not a T_1, T_2-cut, proceed recursively on the graphs G_1 and G_2 obtained by contracting respectively U and $\bar{U}$ into a single T_1-vertex. The uncrossing lemma implies that a minimum T_1, T_2-cut in G either corresponds to a minimum T_1, T_2-cut in G_1 or a minimum T_1, T_2-cut in G_2. Since $|V(G_1)| < |V(G)|$ and $|V(G_2)| < |V(G)|$, eventually the algorithm reaches a base case for which the minimum T_1-cut is also the minimum T_1, T_2-cut. Additionally, $|V(G_1)| + |V(G_2)| = |V(G)| + 2$ and so the implied recursive algorithm runs in polynomial time. $\square$

Generalizations of separating T_1, T_2-cuts are interesting in their own right. In Section 5.2.1, these generalizations and related results are discussed.

4.5 Proof of Lemma 4.3.3

4.5.1 Preliminaries

Let $\mathcal{F}$ be the clutter of odd st-walks of signed graph (G, Σ) with $s, t \in V(G)$. Up to isomorphism, there is exactly one signed graph whose set of odd st-walks is exactly $\mathcal{O}_{K_5}$; odd-K_5 is the signed graph $(K_5, E(K_5))$ where K_5 is the complete graph on five vertices and $s = t$ is any vertex. The correspondence between minors in clutters of odd-st-walks and minors in signed graphs (Proposition 4.2.1) shows that clutter $\mathcal{F}$ has an $\mathcal{O}_{K_5}$ minor if and only if (G, Σ) has an odd-K_5 minor. Up to isomorphism, there is exactly one signed graph whose set of odd st-walks is exactly $\mathcal{L}_7$; (L_7, Σ_7) is the signed graph formed by taking the cycle on 4 vertices with all edges even and adding three odd edges each parallel to a distinct edge of the cycle. Vertices s and t are the endpoints of the edge with no parallel edge (see Figure 4.2). Again, Proposition 4.2.1 implies that clutter $\mathcal{F}$ has no $\mathcal{L}_7$ minor if and only if (G, Σ) has no (L_7, Σ_7) minor.
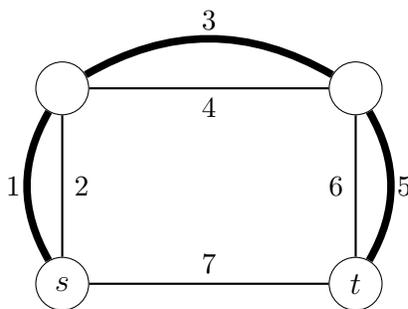
Figure 4.2: Signed graph (L_7, Σ_7). Thick edges are odd.

To prove Lemma 4.3.3, it suffices to show that if $\mathcal{F}$ is Lehman, then we can find $I, J \subseteq E(\mathcal{F})$ such that $(G, \Sigma)/I \setminus J$ is odd-K_5 or $(G, \Sigma)/I \setminus J$ is (L_7, Σ_7). Note that the odd-K_5 and (L_7, Σ_7) we find must be such that the vertices labelled s and t are those vertices previously indicated.

Recall that an element in $b(\mathcal{F})$ is either an st-bond of G or a signature of (G, Σ). By the hypothesis of the Lemma, we have the minimum cardinality sets $\bar{\mathcal{F}}$ of $\mathcal{F}$. We can construct clutter $\bar{\mathcal{K}}$ by solving equation (3.1). Hence throughout the proof, the clutter of minimum cardinality odd-st-walks and the clutter of minimum cardinality st-bonds and signatures are known. The proof has two parts: (1) when $s = t$ in G, we aim to find odd-K_5, and (2) when $s \neq t$ in G, we aim to find (L_7, Σ_7).

4.5.2 Algorithm for Case 1: $s = t$ in G.

As discussed in Section 2.4.2, $\mathcal{F}$ is the clutter of odd circuits of (G, Σ) and $\mathcal{K}$ is the set of all inclusion-wise minimal signatures of (G, Σ). To emphasize this difference, the elements of $\mathcal{F}$ will be denoted by C and elements of $\mathcal{K}$ will be denoted by B throughout this section.

A graph H is a *pseudo-odd-K_5* if there exist $e = xy \in E(H)$, a partition $\{x, y\}, V_1, V_2, V_3$ of $V(H)$ and paths $P_1, P_2, P_3, Q_{12}, Q_{13}, Q_{23}$ such that

i. V_1, V_2, V_3 are stable sets of G,
ii. P_1, P_2, P_3 are pairwise internally vertex disjoint,
iii. for all $i \in [3]$, P_i is between x and y and $V(P_i) \subseteq \{x, y\} \cup V_i$,
iv. for all $i, j \in [3]$, $i \neq j$, Q_{ij} joins $V(P_i)$ to $V(P_j)$, and $V(Q_{ij}) \subseteq V_i \cup V_j$.

We will proceed in two steps: first we show how to use $\bar{\mathcal{F}}$ and $\bar{\mathcal{K}}$ to find a pseudo-odd-K_5, then we show how to obtain an odd-K_5 from a pseudo-odd-K_5. The correctness essentially follows from the proof of Theorem 12 in [13] and will be given in Section 4.5.4.

Finding a pseudo-odd-K_5.

Pick an arbitrary edge $e = xy$ and choose $C_1, C_2, C_3 \in \bar{\mathcal{F}}$ and $B_1, B_2, B_3 \in \bar{\mathcal{K}}$ as in Lehman Corollary 3.4.6. For $i \in [3]$, let P_i denote $C_i - e$; i.e. P_i is a path between x and y. Moreover, it can be shown that property (ii) holds for P_1, P_2, P_3. Since B_1, B_2, B_3 are signatures, $\Sigma' = B_1 \triangle B_2 \triangle B_3$ is a signature of (G, Σ). Contract all edges in $E(G) - \Sigma'$ so that $E(G) = \Sigma'$. For all $i, j \in [3]$, $i \neq j$, $B_i \triangle B_j = (B_i \cup B_j) - e$ is a cut of G since B_i, B_j are signatures. Thus, we can find $U_{ij} \subseteq V(G) - \{x, y\}$ such that $\delta_G(U_{ij}) = B_i \triangle B_j$. For all distinct $i, j, k \in [3]$ we define V_i as $U_{ij} \cap U_{ik}$. Then it can be shown that $\{x, y\}, V_1, V_2, V_3$ is a partition of $V(G)$ and that both properties (i) and (iii) hold. Moreover, it can also be shown that paths Q_{12}, Q_{13}, Q_{23} satisfying property (iv) must exist. Thus (G, Σ') is a pseudo-odd-K_5. Paths Q_{ij} can be found efficiently as it is simply a matter of finding a path between two disjoint sets of vertices in the same component.

Finding an odd-K_5

It remains to show that given a pseudo-odd-K_5 we can either find a pseudo-odd-K_5 with fewer vertices, or we already have an odd-K_5. The argument follows the proof of Lemma 4.1 in [13]. First note that given a pseudo-odd-K_5 we may find Q_{ij} such that only the endpoints intersect P_i and P_j (by possibly taking a subpath of the given path). Given a pseudo-odd-K_5, delete all edges outside of $P_1 \cup P_2 \cup P_3 \cup Q_{12} \cup Q_{13} \cup Q_{23} \cup \{e\}$. Remove

all degree two vertices v by contracting edges of $\delta(v)$. The resulting graph is necessarily also a pseudo-odd-K_5. Moreover, each path P_i contains exactly one internal vertex v_i and since there are no degree two vertices $|V_1| = |V_2| = |V_3|$. We may assume $|V_1| > 1$ for otherwise we already have an odd-K_5. For all distinct $i, j \in [3]$, let e_{ij} be the edge of Q_{ij} that is incident with v_i, let $Q'_{ij} = Q_{ij} - \{e_{ij}, e_{ji}\}$, and let $V'_i = V(Q'_{ij}) \cap V(Q'_{ik})$. Define $G' = G \setminus \{e_{13}, e_{32}, e_{21}\}/\{e_{12}, e_{23}, e_{31}\}$. Necessarily G' is a pseudo-odd-K_5 with partition $\{x, y\}, V'_1, V'_2, V'_3$ and paths $P_1, P_2, P_3, Q'_{12}, Q'_{13}, Q'_{23}$ with fewer vertices than the original pseudo-odd-K_5. Applying this reduction at most $|V_1|$ times, we find an odd-K_5.

4.5.3   Algorithm for Case 2: $s \neq t$ in G.

We look for an (L_7, Σ_7) minor in (G, Σ). The correctness essentially follows from the proof of Theorem 1.1 in [19] and will be given in Section 4.5.5.

It can be shown that there exists an edge $e = st$ and vertex disjoint odd circuits C_s, C_t such that $C_s \cup \{e\}, C_t \cup \{e\} \in \bar{\mathcal{F}}$ and $s \in V(C_s), t \in V(C_t)$. Moreover, there exists an odd circuit C such that $C \cup \{e\} \in \bar{\mathcal{F}}$ and where C and C_s share exactly vertex $v_s \neq s$ and where C and C_t share exactly vertex $v_t \neq t$. It is clear that given $\bar{\mathcal{F}}$, we can find C_s, C_t and C as described above. Deleting all the edges outside of $C_s \cup C_t \cup C \cup \{e\}$ and shortening C_s, C_t and C to two edges by contracting all edges that do not have both ends in $\{s, t, v_s, v_t\}$ yields (L_7, Σ_7).

4.5.4   Proof for Case 1: $s = t$ in G

This proof follows the proof in [28] with the exception that the contraction step occurs earlier.

Finding a pseudo-odd-K_5

We first show that any two minimum cardinality odd circuits of (G, Σ) intersecting in one edge intersect exactly in the edge and its endpoints.

Claim 4.5.1. *Let $C_1, C_2 \in \bar{\mathcal{F}}$. If $e = xy \in C_1 \cap C_2$, then $V(C_1) \cap V(C_2) = \{x, y\}$.*

Proof. By way of contradiction, suppose there exists $v \in V(C_1) \cap V(C_2) - \{x, y\}$. Then there exists $P \subseteq C_1 \cup C_2$ which is neither $C_1 - e$ or $C_2 - e$. By Lehman Corollary 3.4.7, $C_1 \cup C_2 - e$ contains no odd circuit. It follows that P must have odd parity. However, then

$P \cup e$ is an odd circuit in $C_1 \cup C_2$ that is distinct from C_1 and C_2, contradicting Lehman Corollary 3.4.7. $\qquad\square$

As in the algorithm, pick an arbitrary edge $e = xy$ and find C_1, C_2, $C_3 \in \bar{\mathcal{F}}$ and B_1, B_2, $B_3 \in \bar{\mathcal{K}}$ using Lehman Corollary 3.4.6. For the remainder of this analysis, i, j, k denote distinct indices from $\{1, 2, 3\}$.

By Claim 4.5.1, if $P_i = C_i - e$, then P_1, P_2 and P_3 are pairwise internally vertex disjoint paths; property (ii) holds as claimed. Since B_1, B_2 and B_3 are signatures, $B_2 \triangle B_3$ is a cut and thus $B_1 \triangle B_2 \triangle B_3 = B_1 \cup B_2 \cup B_3$ is a signature of (G, Σ'). Let (G', Σ') be obtained from G by contracting all the edges in $E(G) - \Sigma'$ (this is allowed since $E(G) - \Sigma'$ contains no odd circuit as Σ' is a signature). Note that (G', Σ') is $(G', E(G'))$ and any signature of (G', Σ') is also a signature of (G, Σ).

Necessarily B_1, B_2 and B_3 are signatures of (G', Σ') and so $B_i \triangle B_j = B_i \cup B_j - e$ is a cut of G'; there exists $U_{ij} \subseteq V(G') - \{x, y\}$ such that $\delta_{G'}(U_{ij}) = B_i \triangle B_j$. The edges of $E(G') - e$ are spanned by $\delta(U_{12}) \cup \delta(U_{13}) \cup \delta(U_{23})$. Furthermore U_{ij} necessarily induces a connected subgraph of G. Otherwise, there exists $\emptyset \subset T \subset U_{ij}$ with $\delta(T) \subset \delta(U_{ij})$. Then $\delta(T) \triangle B_j \subseteq B_i \triangle B_j$ is a signature of G' and thus a signature of G contained in $B_i \cup B_j$ that is neither B_i nor B_j, contradicting Lehman Corollary 3.4.7. Given B_i, B_j, we can find $\overline{U_{ij}}$ efficiently using breath first search in $G \setminus (B_i \triangle B_j)$ starting from x.

Now, $\delta(U_{12} \triangle U_{23} \triangle U_{13}) = \emptyset$. Since G is connected, $U_{12} \triangle U_{23} \triangle U_{13}$ must be $\emptyset$ since $x, y \notin U_{ij}$. So $U_{ij} = V_i \cup V_j$ where V_1, V_2, V_3 are pairwise disjoint sets of $V \setminus \{x, y\}$. It is clear that $E(G') - e$ is spanned by $\delta(V_1 \cup V_2) \cup \delta(V_1 \cup V_3) \cup \delta(V_2 \cup V_3)$ and $V(G') - \{x, y\}$ is spanned by $V_1 \cup V_2 \cup V_3$. In other words, V_1, V_2, V_3 are independent sets spanning $V \setminus \{x, y\}$. So, we have shown property (i) holds.

Since $E_G(P_i)$ does not contain any edge of $B_j \triangle B_k$, the vertices of P_i are disjoint from $V_j \cup V_k$ in G'. Since $|P_i \cap B_i| \geq 3$, P_i must be a length 2 path between x and y with internal vertex $v_i \in V_i$. So, we have shown properties (ii) and (iii) hold. Lastly, since U_{ij} induces a connected subgraph of G', there must exist a paths Q_{ij} satisfying property (iv).

Finding an odd-K_5

Given a pseudo-odd-K_5 $(G, E(G))$, delete all edges outside of $P_1 \cup P_2 \cup P_3 \cup Q_{12} \cup Q_{13} \cup Q_{23} \cup e$ and remove all degree two vertices by contracting $\delta(v)$. Necessarily $V_1 = V(Q_{12}) \cap V(Q_{13})$, $V_2 = V(Q_{12}) \cap V(Q_{23})$, $V_3 = V(Q_{23}) \cap V(Q_{13})$ and so $|V_1| = |V_2| = |V_3|$. If $|V_1| = 1$, we have found an odd-K_5 minor of (G, Σ) because (G, Σ) is isomorphic to odd-K_5.

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Otherwise, for distinct $\{i, j\} \in \{1, 2, 3\}$ let e_{ij} be the edge on Q_{ij} incident with v_i. Let $G' = G/\{e_{12}, e_{23}, e_3\} \setminus \{e_{13}, e_{32}, e_{21}\}$, $P'_i = P_i$, $Q'_{ij} = Q_{ij} \setminus \{e_{ij}, e_{ji}\}$, $V'_i = V(Q'_{ij}) \cap V(Q'_{ij})$, and $V'_0 = \{x, y\}$. It can be verified that $(G', E(G'))$ is a minor of $(G, E(G))$ that is also a pseudo-odd-K_5. Additionally, $|V'_1| < |V_1|$. Continue inductively to find the odd-K_5 minor. Since the inductive step is applied $|V_1| < n$ times and all the graph operations are implementable in polynomial time, odd-K_5 can be found in polynomial time.

4.5.5   Proof for Case 2: $s \neq t$ in G

This argument follows the proof in [19].

Call an odd circuit C *minimum* if there exists an even path P such that $C \cup P \in \bar{\mathcal{F}}$. Analogously, call an even st-path P *minimum* if there exists an odd circuit C such that $C \cup P \in \bar{\mathcal{F}}$. Call an odd st-path P *minimum* if $P \in \bar{\mathcal{F}}$. To prove correctness of the algorithm, it suffices to show the existence of C_s, C_t and C as described by the algorithm.

Claim 4.5.2. *There exists a minimum odd circuit C_s containing s and a minimum odd circuit C_t containing t. Furthermore, any minimum odd circuit containing s and any minimum odd circuit containing t must be vertex disjoint.*

Proof. Since $\delta(s)$ is an st-cut, it contains some $K \in \bar{\mathcal{K}}$ since it necessarily intersects every (odd or even) st-path. Accordingly, there exists $B \subseteq \delta(s)$ such that $|B \cap F| \geq 1$ for all $F \in \bar{\mathcal{F}}$. By Lehman Corollary 3.4.5, B intersects some $F \in \bar{\mathcal{F}}$ at least twice. This F cannot be an odd st-path and thus it must be the union of an even st-path and an odd circuit. Thus there exists a minimum odd circuit C_s using s. Analogously, there exists a minimum odd circuit C_t using t.

Let C_s be any minimum odd circuit containing s and let C_t be any minimum odd circuit containing t. Suppose C_s uses t. Then C_s can be partitioned into st-paths P and P' one of which must be odd. This contradicts the fact $\mathcal{F}$ is a clutter. Thus C_s cannot use t. Suppose C_s uses some vertex v on C_t where $v \neq t$. Then C_s can be partitioned into sv-paths P_s and P'_s. We may assume P_s is odd and P'_s is even since C_s is odd. Similarly, C_t can be partitioned into tv-paths P_t and P'_t. We may assume P_t is odd and P'_t is even. Then $P_s \cup P'_t$ must contain an odd st-walk F. Since F cannot be the odd st-walk corresponding to C_s or the odd st-walk corresponding to C_t, this is a contradiction to Lehman Corollary 3.4.7. Therefore C_s and C_t are vertex disjoint. $\qquad \square$

Henceforth C_s and will be a fixed minimum odd circuit containing s and C_t will be a fixed minimum odd circuit containing t.

Claim 4.5.3. *There exists a unique minimum st-path P_{even}.*

Proof. Let P_s be the minimum even st-path such that $P_s \cup C_s \in \bar{\mathcal{F}}$. Let P_t be the minimum even st-path such that $P_t \cup C_t \in \bar{\mathcal{F}}$. Then $P_s = P_t$ because otherwise $P_s \cup C_t$ contains an odd st-walk that is neither $P_s \cup C_s$ or $P_t \cup C_t$, a contradiction. This analysis holds for any for $F = P \cup C \in \bar{\mathcal{F}}$ such that $C \neq C_s$. $\square$

Claim 4.5.4. *Let $F \in \bar{\mathcal{F}}$ share vertex $v \neq s$ with C_s. Then either*

1. *F is an odd st-path and uses an edge of C_s incident to s, or,*
2. *F is of the form $P_{even} \cup C$ where C is a minimum odd circuit. Moreover, if C shares another vertex v' with C_s as well, C and C_s are not edge disjoint.*

Proof. By Claim 4.5.3, F either an odd st-path or of the form $P_{even} \cup C$ where C is a minimum odd circuit.

Suppose F is an odd-st-path. Partition circuit C_s into sv-paths Q (odd) and Q' (even). The vertex v splits F into a path F_1 from s to v and a path F_2 from v to t; exactly one of F_1 and F_2 is even. By way of contradiction, suppose F_1 is distinct from Q and Q'. If F_2 is odd, $Q' \cup F_2$ is an odd st-path contained in $F \cup C_s$ that is neither F nor $C_s \cup P_{even}$, contradicting Lehman Corollary 3.4.7. If F_2 is even, $Q \cup F_2$ is an odd st-path contained in $F \cup C_s$ that is neither F nor $C_s \cup P_{even}$, again contradicting Lehman Corollary 3.4.7. Thus F_1 is not distinct from Q and Q' and so F uses an edge of C_s incident to s.

Suppose F is of the form $P_{even} \cup C$ where C is a minimum odd circuit. If C additionally shares vertex v' with C_s, we must show C and C_s are not edge disjoint. By way of contradiction, suppose C and C_s are edge disjoint. Then v and v' partition C_s into P (odd) and P' (even) and v and v' partition C into Q (odd) and Q' (even). All of P, P', Q, Q' must be distinct since C and C_s are edge disjoint. Then $P_{even} \cup P \cup Q'$ contains an odd st-walk distinct from F and $P_{even} \cup C_s$, contradicting Lehman Corollary 3.4.7. $\square$

Assume $F_1 = P_{even} \cup C_s$ and let B_1 denote the mate of F_1.

Claim 4.5.5. *The edges in $F_1 \cap B_1$ are the 2 edges of C_s incident to s and one edge of P_{even}. Thus $|F_1 \cap B_1| = q = 3$.*

Proof. By way of contradiction, suppose there exists $e \in (F_1 \cap B_1) \setminus \delta(s)$. By Lehman Corollary 3.4.3, there exists $F_2 \in \bar{\mathcal{F}}$ such that $F_1 \cap F_2 = \{e\}$. By Claim 4.5.3, F_2 is an st-path because since otherwise F_1 and F_2 share both e and P_{even}. By Claim 4.5.4, since F_2 is an st-path it must use an edge of C_s incident to s. However, e is not incident with s and thus we have a contradiction. We conclude $F_1 \cap B_1 \subseteq \delta(s) \cup P_{even}$ and, moreover, since $\mathcal{F}$ is binary and $|B_1 \cap (P_{even} \cup C_t)| = 1$, $|F_1 \cap B_1| = 3$ and $|B_1 \cap P_{even}| = 1$. $\square$

Let $e_s \in B_1 \cap P_{even}$ and obtain $\{F_1' \ldots F_\ell'\} \in \bar{\mathcal{F}}$ intersecting in at most e_s using Lehman Corollary 3.4.1 where $F_1' = F_1$. Note that there exists $v \in C_s$ since C_s is not a loop by Claim 4.5.5.

Claim 4.5.6. *Let $v \neq s \in V(C_s)$. Then there exists $F' \in \{F_2', \ldots F_\ell'\}$ of the form $P_{even} \cup C$ where $v \in V(C)$.*

Proof. By Claim 4.5.4, no odd-st-path in $\{F_2', \ldots F_\ell'\}$ uses an edge in $\delta(v)$ since otherwise such a path would also intersect F_1 twice: in e_s and in some edge $e \in \delta(s) \setminus e_s$. If there does not exist F' of the form $P_{even} \cup C$ with $v \in V(C)$, since $E(G) = F_1' \cup \ldots \cup F_\ell'$, by Lehman Corollary 3.4.5, v must have degree exactly 2. If $\delta(v) = \{e, e'\}$, every odd st-walk uses both or none of e, e' and so the columns of $M(\bar{\mathcal{F}})$ indexed by e, e' are identical. This contradicts the fact $M(\bar{\mathcal{F}})$ is nonsingular. Thus there exists F' of the form $P_{even} \cup C$ where C uses v. $\square$

Claim 4.5.7. *P_{even} consists of a single edge e_s.*

Proof. If $|P_{even}| \geq 2$, all sets of $\{F_2', \ldots F_\ell'\}$ are odd st-paths, contradicting Claim 4.5.6. $\square$

Now, odd st-walk F' is in $\{F_2', \ldots F_\ell'\}$ if and only if e_s is contained in both F' and its mate. By Claim 4.5.5, there are exactly 3 such sets: $C_s \cup e_s$, $C_t \cup e_t$ and $C \cup e_t$ for some other odd circuit C. Since necessarily C_s and C_t are vertex disjoint, for any vertex $v_s \in V(C_s)$ and $v_t \in V(C_t)$ it must be that $v_s, v_t \in C$ by Claim 4.5.6. Therefore, we showed there exist vertex disjoint odd circuits C_s, C_t and $e = st$ such that $C_s \cup \{e\}$ and $C_t \cup \{e\} \in \bar{\mathcal{F}}$ and where $s \in V(C_s), t \in V(C_t)$. Moreover, C is such that $C \cup \{e\} \in \bar{\mathcal{F}}$ and C and C_s share exactly one vertex $v_s \neq s$ and C and C_t share exactly one vertex $v_t \neq t$. This concludes the proof of correctness for the given algorithm.

4.6   Proof of Lemma 4.3.5

Even though this algorithm is essentially a constructive proof of the version in [19], it was necessary to change the organization of the proof and so the below exposition differs from that in the original paper.

4.6.1 Preliminaries

Let $\mathcal{F}$ be the clutter of st-T-cuts of (G, T) with $s, t \in V(G)$. Up to isomorphism, there is exactly one graft whose set of st-T-cuts is exactly $b(\mathcal{O}_{K_5})$; (O_5, T_5) is the graft in Figure 4.3. The correspondence between minors in clutters of st-T-cuts and minors in grafts (Proposition 4.2.2) shows that clutter $\mathcal{F}$ has a $b(\mathcal{O}_{K_5})$ minor if and only if (G, T) has a (O_5, T_5) minor. Up to isomorphism, there are exactly two grafts whose sets of st-T-cuts are exactly $\mathcal{L}_7$; (L_1, T_1) and (L_2, T_2) are the grafts in Figure 4.4. Again, Proposition 4.2.2 implies that clutter $\mathcal{F}$ has no $\mathcal{L}_7$ minor if and only if (G, T) has no (L_1, T_1) or (L_2, T_2) minor.
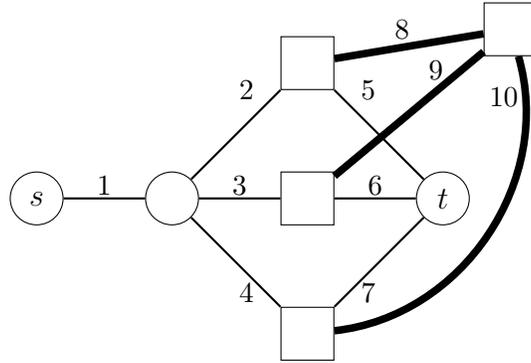
Figure 4.3: Graft (O_5, T_5). Square vertices are in T.
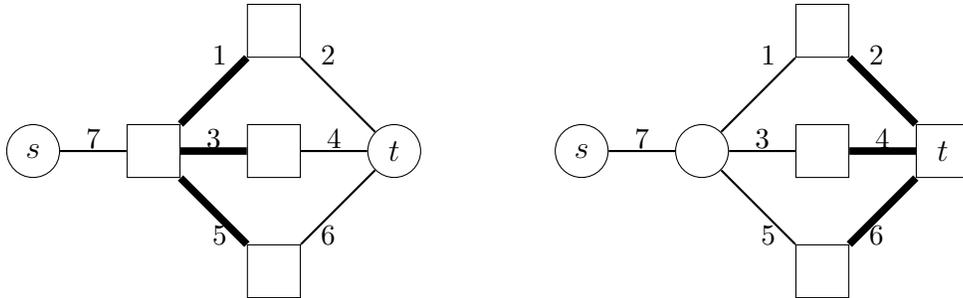
Figure 4.4: Grafts (L_1, T_1) (left) and (L_2, T_2) (right). Square vertices are in T.

To prove Lemma 4.3.5, it suffices to show that if $\mathcal{F}$ is Lehman, we can find $I, J \subseteq E(\mathcal{F})$ such that $(G, T)/J \setminus I$ is (O_5, T_5), (L_1, T_1) or (L_2, T_2).

Recall that an element in $b(\mathcal{F})$ is either an st-path or a T-join (see Section 2.4.5). By the hypothesis of the Lemma, we have the minimum cardinality sets $\bar{\mathcal{F}}$ of $\mathcal{F}$. We can construct clutter $\bar{\mathcal{K}}$ by solving equation (3.1). Hence throughout the proof, the clutter of minimum cardinality odd-st-walks and the clutter of minimum cardinality st-bonds and signatures are both known.

We will refer to elements of $\bar{\mathcal{K}}$ as *minimum T-joins* and *minimum st-paths*. A 3-*path configuration* is a set of st-paths P_1, P_2, P_3 such that

   i. $s \notin T$;
   ii. there exists edge $ss' \in P_1 \cap P_2 \cap P_3$;
   iii. the only vertices common to more than one of P_1, P_2 and P_3 are s, s' and t; and
   iv. each path P_i contains a vertex $v_i \in T$ distinct from s, s' and t.

We define a *link* of (G, T) as a path with both endpoints in $V(P_1) - s$, $V(P_2) - s$, $V(P_3) - s$ and no internal vertices in P_1, P_2 or P_3. We call a link *odd* if is has an odd number of internal vertices in T. We call a link *even* if it has an even number of internal vertices in T.

The algorithm has two main steps. The first is to find minimum st-paths P_1, P_2 and P_3 forming a 3-path configuration. If all T vertices appear on these paths, we are done; delete all edges except $P_1 \cup P_2 \cup P_3$ and contract all edges except ss', $\delta(v_1)$, $\delta(v_2)$ and $\delta(v_3)$. Otherwise, perform contractions and deletions maintaining the 3-path configuration to either push all the T vertices to the paths or find (O_5, T_5).

Recall that r denotes the cardinality of sets in $\bar{\mathcal{F}}$ (the set of minimum st-T-cuts) and ℓ denotes the cardinality of sets in $\bar{\mathcal{K}}$ (the set of minimum st-paths and minimum T-joins). We break the algorithm into two cases: $\ell \geq 4$ and $\ell = 3$. Much of the analysis carries through in both cases. For $\ell = 3$, st-paths in the 3-path configuration do not have an edge incident with any of $\{s, s', t\}$ and so we must reason directly about the structure after some common preliminaries.

4.6.2   Algorithm for Case 1: $\ell \geq 4$

Finding P_1, P_2 and P_3.

Choose $e = ss'$ such that $s \notin T$ and $s' \neq t$. If no such e exists, switch the role of s and t and we are guaranteed that such an edge exists. Use Lehman Corollary 3.4.6 to find minimum paths P_1, P_2, P_3 intersecting exactly in e whose mates B_1, B_2, B_3 intersect exactly in e. If $r = 3$, then all the T vertices are located on the 3-path configuration P_1, P_2, P_3; if v_i is a

T vertex of P_i, deleting all edges except $P_1 \cup P_2 \cup P_3$ and contracting all remaining edges except ss' and $\delta(v_i)$ for $i = 1, 2, 3$, yields either (L_1, T_1) or (L_2, T_2).

Pushing T vertices to paths.

Let edge $ww' \in P_1 - \big(\delta(s) \cup \delta(s') \cup \delta(t)\big)$; such edge exists since $r \geq 4$. Using Lehman Corollary 3.4.6 , find minimum T-joins K_1, K_2 intersecting in exactly ww' whose mates intersect in exactly ww'. Then $C = K_1 \triangle K_2$ is a circuit. If both $s', t \in V(C)$, then all the T vertices are located on the 3-path configuration P_1, P_2, P_3 and we are done. Otherwise, either $s' \notin T$ or $t \notin T$.

Partition edges of $C - (P_1 \cup P_2 \cup P_3)$ into a collection $\mathcal{S}$ of links. Vertices of T are necessarily either on a link of $\mathcal{S}$ or in $P_1 \cup P_2 \cup P_3$. For every $Q \in \mathcal{S}$, contract all edges of Q except for the two edges incident to the ends of Q. The parity of Q remains unchanged; if Q is odd then the internal vertex of Q is in T and if Q is even then the internal vertex of Q is outside T. Update $\mathcal{S}$ by removing all even links. For any pair $S_1, S_2 \in \mathcal{S}$ sharing endpoint v, contract edge of S_1, S_2 incident to v and remove S_1, S_2 from $\mathcal{S}$. If link $S \in \mathcal{S}$ has endpoint $v \in V(P_i) - \{s, s', t\}$ such that $v \notin T$ or there exists $v' \in T - \{s', t, v\}$ in $V(P_i)$, then contract the edge of S incident to v and update $\mathcal{S}$ by removing S.

If after applying this procedure, $\mathcal{S} = \emptyset$, then all the T vertices are located on the 3-path configuration P_1, P_2, P_3 and as before find (L_1, T_1) or (L_2, T_2). Otherwise $\mathcal{S}$ consists of a single odd link with T vertex y. Either there exists paths Q_1, Q_2, Q_3 from y to each of v_1, v_2, v_3 that use no vertex of $P_1 \cup P_2 \cup P_3$ or there exists a path Q from v to s' or t. In the first case, deleting all edges except $P_1 \cup P_2 \cup P_3$ and contracting all remaining edges except ss', $\delta(v_i)$ for $i = 1, 2, 3$ and $\delta(y)$ yields (O_5, T_5). In the second case, deleting all edges except $P_1 \cup P_2 \cup P_3 \cup Q$ and contracting all remaining edges except ss' and $\delta(v_i)$ for $i = 1, 2, 3$ yields (L_1, T_1) or (L_2, T_2).

4.6.3   Proof of Correctness for Case 1: $\ell \geq 4$

Finding P_1, P_2 and P_3.

This proof essentially follows from the proof of Theorem 1.3 in [19]; specifically, we require the following claims. Proofs are included for completeness.

Claim 4.6.1. *Let $K_1, K_2 \in \mathcal{K}$ be two minimum T-joins. Then $K_1 \triangle K_2$ is a circuit that does not use both s and t.*

Proof. Since K_1 and K_2 are T-joins, the vertices of $G[K_1 \triangle K_2]$ have even degree. Thus $K_1 \triangle K_2$ can be expressed as the disjoint union of some set of cycles $\mathcal{C}$ of G. Let $C \in \mathcal{C}$. Then $K_1 \triangle C$ contains a T-join and by Lehman Corollary 3.4.7, $K_1 \triangle C$ must be either K_1 or K_2. Necessarily $K_1 \triangle C \neq K_1$ and so $K_1 \triangle C = K_2$, or equivalently, $K_1 \triangle K_2 = C$. Suppose C contains both s and t. Since K_1 and K_2 intersect in exactly one edge, $|K_1 \triangle K_2| = 2\ell - 2$ and thus C contains an st-path of cardinality at most $\ell - 1$. This contradicts the fact the minimum cardinality elements of $\bar{\mathcal{K}}$ have size ℓ. Therefore, $K_1 \triangle K_2$ is a circuit that does not use both s and t. $\qquad\square$

If P is a path and $x, y \in V(P)$, let $P[x, y]$ denotes the subpath of P between x and y.

Claim 4.6.2. *Let $K_1, K_2 \in \mathcal{K}$ be two minimum st-paths. Then if $v \in V(K_1) \cap V(K_2)$, $K_1[s, v] = K_2[s, v]$ or $K_1[v, t] = K_2[v, t]$. In particular, if K_1 and K_2 intersect in exactly e, then $e \in \delta(s) \cup \delta(t)$.*

Proof. The set $K_1[s, v] \cup K_2[v, t]$ contains an st-path. Lehman Corollary 3.4.7 implies that this path is either K_1 or K_2. The result follows. $\qquad\square$

Claim 4.6.3. *Let $e \in E - \delta(s) - \delta(t)$. Then there exist two minimum T-joins K_1 and K_2 intersecting exactly in e. Endpoints of e that are not in circuit $C = K_1 \triangle K_2$ are in T. At most one endpoint of e is a vertex of C and thus at least one endpoint of e is in T.*

Proof. By Lehman Corollary 3.4.6, there exist $K_1, K_2, K_3 \in \bar{K}$ intersecting exactly in e. Applying Claim 4.6.2, we may assume K_1 and K_2 are T-joins. Let $e = xy$, If $x \notin C$, then $K_1 \triangle \delta(x) = \{e\}$ and thus $x \in T$. If $x, y \in C$, by Lehman Corollary 3.4.6, there exist st-T-cuts $F_1, F_2, F_3 \in \bar{\mathcal{F}}$ intersecting in e. Since F_1, F_2, F_3 are cuts and $x, y \in C$, it must be that $|C \cap F_i| \geq 2$ for all $i \in [3]$. It follows that each of F_1, F_2, F_3 intersect at least one of K_1, K_2 twice. This is a contradiction since at most two of F_1, F_2 can be mates of K_1, K_2. $\qquad\square$

Claim 4.6.4. *No minimum T-join intersects both $\delta(s)$ and $\delta(t)$.*

Proof. Suppose $K \in \bar{\mathcal{K}}$ contains an edge incident to s and an edge incident to t. Let F be the mate of K. Suppose there exists $e \in (K \cap F) - \delta(s) - \delta(t)$. By Lehman Corollary 3.4.6, there exists $K_1, K_2 \in \bar{\mathcal{K}}$ such that K, K_1 and K_2 intersect exactly in e. By Claim 4.6.3, we may assume K_1 is a minimum T-join. Then $K \triangle K_2$ uses both s and t, contradicting Claim 4.6.1. Since $|J \cap U| \geq 3$, we may assume $\{e_s, e'_s, e_t\} \subseteq J \cap U$ for $e_s, e'_s \in \delta(s)$ and $e_t \in \delta(t)$. Since $r \geq 3$, there exists $e \in E - \delta(s) - \delta(t)$. Minimum T-joins K'_1, K'_2 intersecting in exactly e must exist and so there exists minimum T-join K'_1 that does not contain e_t. By

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Claim 4.6.1, $K \triangle K_1'$ is a circuit using one of s and t; by construction, $K \triangle K_1'$ uses t but not s. Since K_1 is not the mate of F, it does not contain both e_s and e_s'. This contradicts the fact $K \triangle K_1$ does not use s. $\qquad \square$

Claim 4.6.5. *Minimum T-joins are either disjoint from $\delta(s)$ or disjoint from $\delta(t)$.*

Proof. Suppose $K_s, K_t \in \bar{\mathcal{K}}$ are such that $K_s \cap \delta(s) \neq \emptyset$ and $K_t \cap \delta(t) \neq \emptyset$. By Claim 4.6.4 $K_s \cap \delta(t) = \emptyset$ and $K_t \cap \delta(s) = \emptyset$. Thus $K_s \triangle K_t$ is a circuit intersecting both s and t, a contradiction to Claim 4.6.1. $\qquad \square$

Claim 4.6.6. *We can find a 3-path configuration.*

Proof. Suppose minimum T-joins are disjoint from $\delta(s)$ and thus $s \notin T$. Let $ss' \in \delta(s)$. By Lehman Corollary 3.4.6, there exist $K_1, K_2, K_3 \in \bar{\mathcal{K}}$ intersecting in exactly ss'. Since $s \notin T$, K_1, K_2, K_3 must be st-paths. By Claim 4.6.3, K_1, K_2, K_3 pairwise share only vertices s, s', t. For each path K_i, there exists $e_i = x_i y_i$ such that $x_i, y_i \notin \{s, s', t\}$ since K_i has length at least 4. By Claim 4.6.3, at least one of x_i, y_i is in T. It follows that K_1, K_2, K_3 is a 3-path configuration. $\qquad \square$

In summary, this section shows we can find a 3-path configuration by simply choosing $s \notin T$ (swapping roles of s and t if necessary), choosing arbitrary $e \in \delta(s)$ and using Lehman Corollary 3.4.6 to obtain 3 st-paths intersecting in exactly e. Necessarily minimum T-joins may be assumed disjoint from $\delta(s)$.

Pushing T vertices to paths.

The proof in this section is, in spirit, the same as the corresponding proof in [19]. However, we must replace a minimality assumption with a method for explicitly finding the minimal instance.

Claim 4.6.7. *Let (G, T) be a graft that contains a 3-path configuration P_1, P_2, P_3. If all vertices of T are in $P_1 \cup P_2 \cup P_3$, then (G, T) contains a (L_1, T_1) or (L_2, T_2) minor.*

Proof. For $i \in [3]$, contract all edges of P_i except $\delta(v_i)$ where $v_i \in T$. Delete all edges outside $P_1 \cup P_2 \cup P_3$. In the resulting graft, necessarily one of s' and t is a T-vertex and so we have either (L_1, T_1) or (L_2, T_2). $\qquad \square$

Claim 4.6.8. *If $r = 3$, then all vertices of T are in $P_1 \cup P_2 \cup P_3$.*

Proof. By Lehman Corollary 3.4.2, $E = P_1 \cup P_2 \cup P_3$ and so the result follows. □

Let edge $ww' \in P - \delta(s) - \delta(s') - \delta(t)$. Such an edge must exist since $r \geq 4$. By Lehman Corollary 3.4.4, there exists a collection $\mathcal{K}' \subseteq \bar{\mathcal{K}} = \{K_1, K_2 \ldots K_r\}$ intersecting in at most ww'. Using Claim 4.6.3, we may assume $K_1, K_2 \in \mathcal{K}$ intersect in exactly ww' and the mates of K_1 and K_2 also intersect in exactly ww'. By Claim 4.6.5 $C = K_1 \triangle K_2$ is a circuit that does not use s. In fact, we can do better.

Claim 4.6.9. *If C uses both s' and t, then the vertices of T are in $P_1 \cup P_2 \cup P_3$.*

Proof. If C uses both s' and t, then we can partition C into paths Q_1, Q_2 with endpoints s' and t. Since $|Q_1| + |Q_2| = 2\ell - 2$, $Q_1' = Q_1 \cup ss'$ and $Q_2' = Q_2 \cup ss'$ are minimum st-paths (i.e., $Q_1', Q_2' \in \bar{\mathcal{K}}$). Let F_1' be the mate of Q_1' and F_2' be the mate of Q_2'. Since K_1 and K_2 intersect F_1' and F_2' exactly once, $ss' \in F_1'$ and $ss' \in F_2'$. If F_1, F_2, F_3 are the mates of P_1, P_2, P_3, since P_1, P_2, P_3 are a 3-path configuration, $ss' \in F_1$, $ss' \in F_2$, and $ss' \in F_3$. Since $(\mathcal{F}, \mathcal{K})$ is a Lehman blocking pair, nonidentical elements of $\{P_1, P_2, P_3, Q_1', Q_2'\}$ pairwise intersect in exactly ss'. By Claim 4.6.2, nonidentical elements only share the vertices $\{s, s', t\}$. Since the vertices of $T - \{w, w'\}$ are in C and each P_i has a vertex in $T - \{s, s', t\}$, we can assume that $Q_1' = P_2$ and $Q_2' = P_3$. It follows that the vertices of T are in $P_1 \cup P_2 \cup P_3$. □

Since C does not use both s' and t then either $s' \notin T$ or $t \notin T$.

Claim 4.6.10. *If C does not use both s' and t, then there exists a minimum T-join $K \in \mathcal{K}' - \{K_1, K_2\}$.*

Proof. By way of contradiction, suppose $K_3 \ldots K_r$ are all st-paths. By Claim 4.6.2, they must be internally vertex disjoint. By Lehman Corollary 3.4.2, $E = K_1 \cup \ldots \cup K_r$. Vertex s' must have degree three because 3-path configuration P_1, P_2, P_3 uses s'. Necessarily K_1 and K_2 use s' and so $s' \in C$ and thus $t \notin C$. Each edge of $\delta(t)$ is on one of $K_3 \ldots K_r$ and since $|\delta(t)| \geq 3$, this implies $r \geq 5$. If v is a degree 2 vertex on one of $K_3 \ldots K_r$, then $v \in T$ or else $M[\bar{\mathcal{K}}]$ is nonsingular, contradicting the fact $(\mathcal{F}, \mathcal{K})$ is a Lehman blocking pair. It follows that the vertices of $K_3 \ldots K_r$ are in $C \cup \{w, w'\}$. Since C has $2\ell - 2$ vertices, it follows that $(r - 2)(\ell - 2) \leq 2r$. Since $r \geq 5$, this implies $\ell = 3$, a contradiction. Thus we may assume $K_3 \in \bar{\mathcal{K}}$ is a minimum T-join. □

Partition $C - (P_1 \cup P_2 \cup P_3)$ into a collection $\mathcal{S}$ of links. Vertices of T are either on a link of $\mathcal{S}$ or in $P_1 \cup P_2 \cup P_3$. If $s' \notin T$, no link has endpoint s'. If $t \notin T$, no link has endpoint t. We say that link $S \in \mathcal{S}$ is *in an $\mathcal{S}$-tripod* if there exists a path Y with endpoints y, y' such that

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1. internal vertices of Y are not in $P_1 \cup P_2 \cup P_3$ or in any $S \in \mathcal{S}$,
2. y is an internal vertex of S that is in T, and
3. y' is not an endpoint of S, $y' \neq s$ and y' is either a vertex of P_1, P_2 or P_3 or an internal vertex of link $S' \in \mathcal{S}$ for $S' \neq S$.

Claim 4.6.11. *If $S \in \mathcal{S}$ is an odd link with both endpoints in T then S is in an $\mathcal{S}$-tripod.*

Proof. Let v, v' be the endpoints of S and $V_s = V(S) \cap T$. Since S is odd, $|V_s|$ is odd. By Claim 4.6.1, $C' = K_1 \triangle K_3$ is a circuit. Thus $K_3 \cap C'$ is a collection $\mathcal{L}$ of pairwise vertex disjoint paths with endpoints in $T \cup \{w, w'\}$ and internal vertices in $V - T - \{w, w'\}$. Every vertex in V_s is an endpoint of some path in $\mathcal{L}$. Let $\mathcal{L}^+ \subseteq \mathcal{L}$ be the set of paths with exactly one endpoint in V_s. Since $|V_s|$ is odd, $|\mathcal{L}^+|$ must be odd.

By way of contradiction, suppose all paths in $\mathcal{L}^+$ have endpoints in $V_s \cup \{v, v'\}$. Then $\mathcal{L}^+ = \{L_+\}$; assume this path has endpoint v. Assume the edge of S incident to v is in K_1 and the edge of S incident to v' is in K_2. Vertices of $V_s \cup v$ are incident to an edge of $K_1 \cap S$ and to either an edge of a path in $\mathcal{L} - L_+$ or an edge of L_+. The edges in $E' = (K_1 \cap S) \cup \mathcal{L} - L_+$ is a disjoint union of circuits. However, $E' \subseteq K_1 \triangle K_3$ and since C' is a circuit containing all vertices in $T - \{w, w'\}$, the vertices of $T - \{w, w'\}$ are vertices of S. Since $v' \in T - C$, either $v' = w$ or $v' = w'$. So v' is on P_1, v is on P_2, but there cannot be a T-vertex on P_3, a contradiction.

So there exists $L \in \mathcal{L}_+$ with endpoint $y \in V_s$ and endpoint y' outside $V_s \cup \{v, v'\}$. Let y' be first vertex of L starting from y which is a vertex of P_1, P_2, P_3 or some link of $\mathcal{S} - S$. The path $Y = L[y, y']$ is an $\mathcal{S}$-tripod since $y' \neq s$ since $s \notin C'$ by Claim 4.6.5. $\qquad \square$

Let (G', T') and $\mathcal{S}'$ be obtained from (G, T) and $\mathcal{S}$ by performing the following operations.

1. For all $S \in \mathcal{S}$, contract all edges of Q except for the two edges incident to the ends of Q.
2. Remove all even links from $\mathcal{S}$.
3. For any pair $S_1, S_2 \in \mathcal{S}$ sharing endpoint v, contract edge of S_1, S_2 incident to v and remove S_1, S_2 from $\mathcal{S}$.
4. If link $S \in \mathcal{S}$ has endpoint $v \in V(P_i) - \{s, s', t\}$ such that $v \notin T$ or there exists $v' \in T - \{s', t, v\}$ in $V(P_i)$, then contract the edge of S incident to v and update $\mathcal{S}$ by removing S.

Since whenever $s' \notin T$, no link has endpoint s', then $s' \notin T'$. Since whenever $t \notin T$, no link has endpoint t, then $t \notin T'$. Moreover, since s is not the endpoint of any link in $\mathcal{S}$, we maintain our 3-path configuration.

Remark 4.6.12. P_1, P_2, P_3 *is a 3-path configuration for* (G', T').

Claim 4.6.13. *If not all vertices of T' are on P_1, P_2, P_3, then $\mathcal{S}'$ contains a unique odd link S' such that for some internal vertex y*

1. *there exists paths Q_1, Q_2, Q_3 from y to each of v_1, v_2, v_3 that uses no vertex of $P_1 \cup P_2 \cup P_3$, or*
2. *there exists a path Q from v to s' or t.*

Proof. This follows from two observations. Firstly, if more than two odd links remain, we can necessarily apply operation 3 or operation 4. Secondly, if an odd link began in an $\mathcal{S}$ tripod, then it is in an $\mathcal{S}'$ tripod throughout the procedure. Thus exactly one odd link remains and since it is an $\mathcal{S}'$ tripod, one of the given cases must hold. $\qquad\square$

In the first case, deleting all edges except $P_1 \cup P_2 \cup P_3$ and contracting all remaining edges except ss', $\delta(v_i)$ for $i = 1, 2, 3$ and $\delta(y)$ yields (O_5, T_5). In the second case, deleting all edges except $P_1 \cup P_2 \cup P_3 \cup Q$ and contracting all remaining edges except ss' and $\delta(v_i)$ for $i = 1, 2, 3$ yields (L_1, T_1) or (L_2, T_2). This concludes the proof of correctness.

4.6.4 Case 2: $\ell = 3$

Here the proof and algorithm are given simultaneously. Since $\ell = 3$, and there exists minimum T-joins in $\bar{\mathcal{K}}$, it must be that $|T| = 2$, $|T| = 4$ or $|T| = 6$.

Find P_1, P_2, P_3.

All the claims regarding finding 3-path configuration for $\ell \geq 4$ hold when $\ell = 3$ except for Claim 4.6.6, In Claim 4.6.6, we required our st-paths to be length 4 to guarantee there was some edge such that neither of the endpoints were s, s' or t.

Claim 4.6.14. *We can find a 3-path configuration.*

Proof. As per proof in $\ell \geq 4$ section, we have paths $P_i = \{ss', s'v_i, v_it\}$ for $i = 1, 2, 3$. If $s' \notin T$, then necessarily $v_i \in T$ for all i and we have a 3-path configuration.

Otherwise, by way of contradiction, suppose $v_1 \notin T$. By Claim 4.6.3, we can find minimum T-joins K_1 and K_2 intersecting in exactly edge $s'v_1$. By Claim 4.6.1, $K_1 \triangle K_2$ is a cycle C and since $\ell = 3$, $|C| = 4$. Since $v_1 \notin T$ and all vertices of $T - \{s'\}$ are vertices

of C, $|T| = 4$. It follows that there exist $w_1, w_2, w_3 \in T$ such that $K_1 = \{v_1 w_1, w_2 w_3, v_1 s'\}$ and $K_2 = \{w_1 w_2, w_3 v_1, v_1 s'\}$. If st-T-cut $\delta(U)$ is the mate of minimum st-path P_1 then we may assume $s, v_1 \notin U$, $s', t \in U$. Since $\delta(U)$ is not the mate of K_1 or K_2, $\delta(U) \cap K_1 = \delta(U) \cap K_2 = \{s' v_1\}$. It follows that $U \cap T = s'$ and so $t \notin T$. Necessarily no minimum T-join contains edge $v_1 t$ since $|T| = 4$. This implies three minimum T-joins intersect in $v_1 t$.

Since $v_1, t \notin T$, we can use edge $v_1 t$ to find our 3-path configuration. It remains to verify all minimum T-joins are disjoint from $\delta(t)$ so we can continue to use this property. Otherwise, some T-join uses two edges in $\delta(t)$ and this contradicts $\ell = 3$ since it implies the existence of a length 2 st-path. $\qquad \square$

In summary, if choosing arbitrary $ss' \in \delta(s)$ such that $s' \in T$ and applying Lehman Corollary 3.4.6 does not lead to a 3-path configuration, then we can still find a 3-path configuration. Simply relabel s and t so that $s \notin T$, no minimum T-join intersects $\delta(s)$ and there exists edge $ss' \in \delta(s)$ such that $s' \notin T$; the above claim guarantees we can do this.

Note that since there exists a 3-path configuration with $\{v_1, v_2, v_3\} \subseteq T$, $|T| = 4$ or $|T| = 6$. There are two cases: $s' \notin T$ and $s' \in T$.

Case 1: $s' \notin T$

Consider edge $s' v_1$. By Lehman Corollary 3.4.4, there exists a collection $\mathcal{K}' \subseteq \bar{\mathcal{K}} = \{K_1, K_2 \ldots K_r\}$ intersecting in at most $s' v_1$. Using Claim 4.6.3, we may assume $K_1, K_2 \in \mathcal{K}$ intersect in exactly $s' v_1$ and the mates of K_1 and K_2 also intersect in exactly $s' v_1$. Since $s' \notin T$, necessarily $|T| = 4$. If $t \in T$, then we have found (L_1, T_1). Otherwise, let $T = \{v_1, v_2, v_3, y\}$. Necessarily K_1 and K_2 are the union of

1. an edge joining two T vertices, and
2. a length 2 path with internal vertex s' joining two T vertices, one of which is v_1.

If the endpoint of the length 2 path in K_1 or K_2 is y, then we are done. Simply contract the edge joining s' and y, and delete all edges outside of $P_1 \cup P_2 \cup P_3$. Otherwise, the end of the length 2 path in K_1 is v_2 and the end of the length 2 path in K_2 is v_3. This implies there is an edge joining v_2 and y and an edge joining v_3 and y.

In a similar way, there are minimum T-joins K_1' and K_2' intersecting in at most $s' v_2$ and of the form above. If the endpoint of the length 2 path of K_1 or K_2 is y, then we are done. Otherwise, we are guaranteed there are edges e_1, e_2, e_3 joining y to all of $\{v_1, v_2, v_3\}$. Deleting all edges except $P_1 \cup P_2 \cup P_3 \cup \{e_1, e_2, e_3\}$ yields (O_5, T_5).

Case 2: $s' \in T$

If $|T| = 4$, then deleting all edges except $P_1 \cup P_2 \cup P_3$ yields (L_1, T_1). Thus, we consider the case when $|T| = 6$. Suppose $T = \{s, v_1, v_2, v_3, x, y\}$ where x or y is possibly t. If x or y is t, Claim 4.6.9 shows all T vertices are on $P_1 \cup P_2 \cup P_3$. This is, however, impossible since $s \notin T$ and $|V(P_1 \cup P_2 \cup P_3)| = 6$. By Lehman Corollary 3.4.4, there exists a collection $\mathcal{K}' \subseteq \bar{\mathcal{K}} = \{K_1, K_2 \ldots K_r\}$ intersecting in at most $s'v_1$. Using Claim 4.6.3, we may assume $K_1, K_2 \in \mathcal{K}$ intersect in exactly $s'v_1$ and the mates of K_1 and K_2 also intersect in exactly $s'v_1$. Since $|T| = 6$, $s, v_1 \in T$, $K_1 = \{sv_1, e_1, e_2\}$ and $K_2 = \{sv_1, e'_1, e'_2\}$, where $\{e_1, e_2\}$ and $\{e'_1, e'_2\}$ are matchings of $\{v_2, v_3, x, y\}$.

If any of e_1, e_2, e'_1, e'_2 join x and y, then contracting the edge xy and deleting all edges except $P_1 \cup P_2 \cup P_3$ yields (L_1, T_1). Otherwise we may assume $e_1 = v_2x$, $e_2 = v_3y$, $e'_1 = v_2y$, $e'_2 = v_3x$. Contracting $\{e_1, e_2, e'_1, e'_2\}$, and deleting all edges except $P_1 \cup P_2 \cup P_3$ yields (L_1, T_1).

4.7 Remarks

The problem of finding a fixed minor of a binary clutter is equivalent to the problem of finding a rooted minor in a binary matroid. It follows from recent developments in the matroid minor project that this problem can be solved in polynomial time [12]. Using these algorithms, it is possible to avoid using Lemma 4.3.3, Lemma 4.3.5 and Theorem 3.1.3 in the proofs of Theorem 4.1.1 and Theorem 4.1.2.

We have already introduced Corollary 1.3.4 of Theorem 4.1.1: given a graph and weight function on its edges, in polynomial time we can either find a maximum weight cut or find K_5 as an odd minor. Note that this result also follows from the graph minor testing algorithm with parity condition [21].

Proof of Corollary 1.3.4. Given G, let $\Sigma = E(G)$ and label some arbitrary vertex by $s = t$. If $\mathcal{F}$ is the clutter of odd-st-walks of (G, Σ) then it is, in fact, the clutter of odd circuits of G. By Remark 4.3.1 and Proposition 4.3.2 we can, in polynomial time, find an extreme point $\bar{x}$ of $Q(\mathcal{F})$ that is optimal for (2.5) together with a set of n constraints of $Q(\mathcal{F})$ that define $\bar{x}$. If $\bar{x}$ is integer, then we may assume that it is the characteristic vector of a set of edges B that intersects every odd circuit; i.e., $E(G) - B$ is the maximum cut. Otherwise, use Theorem 3.1.3 and Lemma 4.3.3 to find I, J such that $\mathcal{F}/I \setminus J$ is isomorphic to $\mathcal{O}_{K_5}$. It can be then readily checked that I forms a cut of G and that K_5 is obtained by contracting all edges of I and deleting parallel edges. $\square$

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Chapter 5

Future Work, Related Problems and Conclusions

5.1 Future Work: Eulerian Multi-Commodity Flow Algorithms

Recall that the fractional, integer and Eulerian integer flow problems are solved for graphs. That is, we know the minor minimal signed graphs such that (G, Σ) does not admit an integer (or fractional) flow for every (or every Eulerian) weight function. Moreover, polynomial time algorithms for finding flows or obstructions are known for the fractional and integer cases. However, there does not exist a polynomial time algorithm corresponding to the result [13] on Eulerian integer multi-commodity flow.

Theorem 5.1.1 (Theorem 1.1.4)**.** *Suppose (G, Σ) has no odd-K_5 minor. Then for every Eulerian weight function $w \in \mathbb{Z}_+^E$ satisfying the cut condition, (G, Σ) has an integer w-flow.*

We are interested in whether the construction in the above theorem can be used to obtain a polynomial time algorithm for the following problem: given (G, Σ) and Eulerian $w \in \mathbb{Z}_+^E$, either show the cut condition is violated, show (G, Σ) has an odd-K_5 minor, or find an integer w-flow.

We have no satisfactory answer to this question. Here we outline the unsuccessful strategy used and explain why doing the "obvious thing" did not work. In the positive direction, the construction in Theorem 1.1.4 leads to a finite algorithm for finding fractional flows; the general outline of this technique is found in Section 5.1.2. Additionally,

we show that a certain type of incremental flow algorithm running in pseudo-polynomial time and finding either an obstruction or flow can be used to obtain a polynomial time algorithm to do the same. Although this seems like an unnatural construction, were the algorithm of Section 5.1.2 to run pseudo-polynomial time, it would result in exactly the incremental algorithm needed. For more details, see Section 5.1.3. We begin by discussing the transversal property of flows and outline two paradigms for designing flow algorithms: maintaining optimality and maintaining feasibility. The finite time algorithm of Section 5.1.2 maintains optimality while the capacity scaling algorithm of Section 5.1.3 maintains feasibility.

5.1.1   Transversal Property, Underflows and Overflows

Consider the unit weight case. An integer $\mathbb{1}$-flow is a set of edge disjoint odd circuits. A cover is a set of edges intersecting every odd circuit. Let $\mathcal{P}$ be a maximum set of edge disjoint odd circuits and let B be a minimum cover. If (G, Σ) has the Eulerian integer MFMC property, $|\mathcal{P}| = |B|$ and so B is a transversal of $\mathcal{P}$. That is, each circuit in $\mathcal{P}$ contains exactly one edge in B and each edge in B is contained in exactly one circuit of $\mathcal{P}$. Given a maximum set of odd circuits $\mathcal{P}$, finding a minimum cover is equivalent to choosing an edge from each $P \in \mathcal{P}$ so that the chosen edges intersect every odd circuit of (G, Σ). Symmetrically, given a minimum cover, finding an integer $\mathbb{1}$-flow is equivalent to choosing a set of disjoint circuits - one through each edge in B.

In the weighted case, there is an similar transversal property. Let B be some minimum cover of (G, Σ) for Eulerian weight function w. If (G, Σ) has the Eulerian integer MFMC property, then any maximum integer w-flow $y \in \mathbb{Z}_+^{\mathcal{C}_1}$ is such that $\sum(y_C : d \in C \in \mathcal{C}_1) = w_d$ for all $d \in B$. That is, the minimum cover is (in a weighted sense) a transversal of the maximum w-flow. Suppose we are given minimum cover B. Finding a w-flow can be viewed as choosing, for each $d \in B$, w_d circuits intersecting B in exactly $\{d\}$.

Algorithm design paradigms for flow problems fall into two categories: algorithms that maintain optimality and algorithms that maintain feasibility. For example, Ford Fulkerson [11] maintains feasibility while Preflow Push [15] maintains optimality. This dichotomy follows through for multi-commodity flow instances.

We call $y \in \mathbb{Z}_+^{\mathcal{C}_1}$ an *overflow* if for each $d \in \Sigma$, $\sum(y_C : d \in C \in \mathcal{C}) = w_d$; i.e., if it satisfies (1.1). An overflow satisfies the demand constraints but not necessarily the capacity constraints. If $\sum(y_C : e \in C \in \mathcal{C}) = w_e + \Gamma_e$ for $\Gamma_e \geq 1$, $\Gamma_e \in \mathbb{Z}$, we call e an *inflated edge*. The total inflation of overflow $y \in \mathbb{Z}_+^{\mathcal{C}_1}$ is $\Gamma(y) = \sum(\Gamma_e : e \text{ is inflated edge})$. An algorithm maintaining feasibility begins with overflow y having total inflation $\Gamma(y)$ and

tries to construct overflow y' having total inflation $\Gamma(y')$ such that $\Gamma(y') < \Gamma(y)$. Overflows are used in Section 5.1.2 to describe the finite time constructive algorithm arising from Theorem 1.1.4.

We call $y \in \mathbb{Z}_+^{\mathcal{C}_1}$ an *underflow* if for each $e \in E$, $\sum(y_C : e \in C \in \mathcal{C}_1) \leq w_e$; i.e, if it satisfies (1.2). An underflow satisfies the capacity constraints but not necessarily the demand constraints. An algorithm maintaining feasibility works to construct from underflow y, an underflow y' such that $\sum(y_C : C \in \mathcal{C}) < \sum(y_C' : C \in \mathcal{C})$. Since the value of the underflow increases by at least 1 by doing this, eventually a flow will be found. The *deficiency* of underflow y is defined as the difference between the size of the minimum cover and the amount of satisfied demand. The deficiency of underflow y is given by $\Delta(y) = w(\Sigma) - \sum(y_C : C \in \mathcal{C}_1)$. Underflows are used in Section 5.1.3 to show that the existence of an incremental pseudo-polynomial time algorithm finding integer flows or obstructions implies the existence of a polynomial time algorithm doing the same.

In summary, we have three types of flows: *underflows* satisfy capacity constraints; *overflows* satisfy demand constraints; *flows* satisfy demand and capacity constraints.

5.1.2 Finite Time Algorithm Arising from Theorem 1.1.4

Suppose we are given (G, Σ) and Eulerian $w \in \mathbb{Z}_+^E$. The first step is to find a minimum weight signature Σ_w. Note that we can either find such a signature or find an odd-K_5 minor in polynomial time because of Theorem 4.1.1. If we find an odd-K_5 minor, we are done. If $w(\Sigma_w) < w(\Sigma)$, then the cut condition is violated and we are done. Otherwise, we conclude that Σ is indeed a minimum weight signature and thus the cut condition holds. Since the cut condition holds, finding an integer w-flow is equivalent to finding an integer packing of odd circuits of (G, Σ) of size $w(\Sigma)$.

Note although that flow decomposition techniques admit polynomial size representations for overflows, for the purposes of discussion we will simply work with overflows represented by $y \in \mathbb{Z}_+^{\mathcal{C}_1}$.

Generate an overflow by choosing for each $d \in \Sigma$ exactly w_d circuits intersecting Σ in exactly $\{d\}$. If there is no inflated edge, then we already have a flow. Otherwise, there exists an edge $e \in E$ such that $\sum(y_C : e \in C \in \mathcal{C}) = w_e + \Gamma_e$ for $\Gamma_e \geq 2$. The goal is to decrease Γ_e by 2.

First, find circuits C_1, C_2, C_3 using edge e with $y_{C_i} \geq 0$. We update circuits C_1, C_2, C_3 until none are removable. Consider weight function w' with $w_e' = w_e - 1$ for all $e \in C_i$ and $w_e' = w_e$ for all $e \notin C_i$. Circuit C_i is *removable* if the weight of the minimum cover of

G with weight function w is exactly one more than the weight of a minimum cover of G with weight function w'. If C_i is removable, we update our weight function to w' and 'set C_i aside'; applying this reduction until none of C_1, C_2, C_3 are removable suppose we set aside $k(C)$ copies of $C \in \mathcal{C}$. If we terminate in this iteration, we increase y_C by $k(C)$ for all $C \in \mathcal{C}$ to obtain a solution. If we do not terminate in this iteration, we increase y_C by $k(C)$ for all $C \in \mathcal{C}$ to obtain the overflow for the next iteration. After this step, we have some new weight function w' and circuits C_1, C_2, C_3 using e, none of which are removable.

If $w'_e \geq 2$, using the construction in [13], either:

1. *Untangling step*: Obtain circuits C'_1, C'_2, C'_3 such that $E(C'_1 \cup C'_2 \cup C'_3) \subseteq E(C_1 \cup C_2 \cup C_3)$ (counting multiplicities) and $e \notin C'_1$, $e \notin C'_2$.
2. Find odd-K_5 minor.

In the second case, we are finished. In the first case set $y_{C_i} = y_{C_i} - 1$ and $y_{C'_i} = y_{C'_i} + 1$ and continue by choosing some other inflated edge w_e.

If $w'_e \leq 1$, we have to apply the algorithm recursively on $(G - e, \Sigma - e)$ to do the untangling step. Note that if $\Gamma_e = 1$ for some e, we need to untangle in a different way; the ability to do so follows from the fact the original weight function was Eulerian.

Although this algorithm will eventually terminate since we call recursively on increasingly smaller graphs, the recursive calls lead to an exponential running time. The use of removable circuits is not ideal; the main issue is that we have no relationship of the type 'If a circuit is removable for weight function w_1, then it is also removable for related weight function w_2'. If we could guarantee a removable circuit for the original weight function could be located every polynomially many steps, this would in fact imply a pseudo-polynomial time algorithm for the problem.

5.1.3 Capacity Scaling

Suppose we have an algorithm with the following input and output.

- Input:
  1. (G, Σ), and
  2. $w \in \mathbb{Z}_+^E$ such that the cut condition holds, and
  3. underflow y with deficiency $\Delta(y)$
- Output either
  1. w-flow y', or
  2. $I, J \subseteq E(G)$ such that $(G, \Sigma)/I \setminus J$ is odd-K_5

We will call such an algorithm a `packAdditional` algorithm.

The algorithm discussed in Section 5.1.2 is, in fact, a `packAdditional` algorithm. Underflow y provides a mechanism for choosing our starting set of circuits for the algorithm; if $\sum(y_C : d \in C \in \mathcal{C}_1) = z_d$, then we choose an additional $w_d - z_d$ circuits intersecting Σ in exactly $\{d\}$. This algorithm does not run in pseudo-polynomial time in deficiency $\Delta(y)$; however, this had been the initial hope.

A `packAdditional` algorithm running in pseudo-polynomial time would in fact imply a polynomial time algorithm for the problem of finding a w-flow or odd-K_5 for multi-commodity flow instance for which the cut condition holds. This is proven below using a capacity scaling scheme. Note that this scheme requires the ability to calculate minimum covers and thus could not be applied for arbitrary multi-commodity flow instances. Since multi-commodity flow is strongly NP-hard for $|\Sigma| \geq 2$, schemes of this sort would be uninteresting since a pseudo-polynomial time algorithm does not exist unless $P = NP$. Additionally, devising `packAdditional` type algorithms is difficult since it requires a strong understanding as to what types of partial solutions can lead to complete solutions.

The capacity scaling scheme used here is similar to the one used for Ford Fulkerson (see [11]). The main idea is to use the large units of available capacity first and then add bits of accuracy in further iterations. If the cut condition holds for initial weight function w, it may not hold for w' obtained by restricting w to the highest order bit of any entry. Accordingly, we must adjust our demand at each step to guarantee we are still working with an instance satisfying the cut condition. Also note that we must decrease demand on all demand edges since we do not know exactly how to redistribute demand to restore the cut condition otherwise. The key observation that makes the scheme work is that if we choose weights at one iteration such that the cut condition holds, the violation in the next iteration is bounded by some function of $|E|$ and $|\Sigma|$.

In stating the capacity scaling scheme, `packAdditional`(G, Σ, w, y) denotes a call to the `packAdditional` algorithm for signed graph (G, Σ), weight function w, and underflow y. Let W denote the largest entry of w. The capacity scaling algorithm is as follows.

```
Let  k = ⌊log₂ W⌋.
Let  y = 0.
While  k > 0 Do
    Let  w' = ⌊w/2ᵏ⌋.
    Find τ_{w'}, the weight of minimum signature of (G, Σ) for weight function w'.
    Let the excess E be ∑(w'_d : d ∈ Σ) − τ'_w.
    Set w''_d = max(w'_d − E, w_d(2y)).
    y = packAdditional(G, Σ, w'', 2y).
```

End

To analyze the running time of this algorithm, first note that we need to find $\tau_{w'}$. Theorem 4.1.1 of this thesis shows we can efficiently find $\tau_{w'}$ or show odd-K_5 is a minor of (G, Σ). The outer loop is performed $\log_2 W$ times. As long as the deficiency for each `packAdditional` call is polynomial in $|\Sigma|$ and $|E|$, this would imply that the entire algorithm runs in polynomial time. The below lemma shows that the deficiency in each iteration is polynomially bounded.

Lemma 5.1.2. *Let w be the weight function in iteration i of the outer loop and w' be the weight function in iteration $i + 1$ of the outer loop. Then $\sum(w'_d : d \in \Sigma) \leq 2 \sum(w_d : d \in \Sigma) + |\Sigma||E|^2$.*

Proof. If we assume w' satisfies the cut condition (as it would in the last iteration), the analysis is straightforward. If Σ_w is a minimum weight signature of (G, Σ) for weight function w, then $\Sigma_w = \Sigma \triangle \delta(S)$ for $S \subseteq V$. Since $w'(\Sigma_w) = w'(\Sigma \triangle \delta(S)) \geq w'(\Sigma)$ and $w'_e = 2w_e + r_e$ for $r_e \in \{0, 1\}$, $w(\Sigma) - 2w(\Sigma_w) \leq |E|$. In other words, $2\tau_w \geq \tau_{w'} - |E|$ and the excess in each iteration is bounded by $|E|$. Accordingly, the difference in demand between two iterations in bounded by $|E||\Sigma|$ since we subtract the excess from every demand edge. Lastly, we note that if w' does not satisfy the cut condition, the above analysis holds by comparing Σ_w and $\Sigma_{w'}$ instead. $\square$

5.2   Other Related Problems

The Flowing and Cycling Conjectures (Conjectures 1.2.4 and 1.2.3) remain open for many classes of clutters. In Section 2.4.4 and 2.4.5, we showed that for duals of even cut and even cycle matroids, the single commodity Cycling Conjecture implies the Four Colour Theorem. This suggests the conjecture may be difficult to resolve even for the single commodity case.

5.2.1   Cuts with Parity Conditions

The separation problem for st-T-cuts has some generalizations of interest. As mentioned in Section 4.4, one generalization is finding T_1-odd, T_2-odd cuts. That is, given $G = (V, E)$ and $T_1, T_2 \subseteq V$ with $|T_1|, |T_2|$ even, we want to find a minimum weight cut $\delta(S)$ such that $|T_1 \cap S|$ is odd and $|T_2 \cap S|$ is odd. An uncrossing lemma of [16] shows that minimum weight T_1-odd, T_2-odd cuts can be found efficiently. Faster algorithms for this problem have been developed; see [14] and [2]. One related question is whether we can get efficient

algorithms for different parity conditions. A T_1-odd, T_2-even cut is $\delta(S)$ for $S \subseteq V$ such that $|T_1 \cap S|$ is odd and $|T_2 \cap S|$ is even. Using an uncrossing lemma, we can again find minimum T_1-odd, T_2-even cuts efficiently ($\delta(S)$ is T_1-odd and T_2-even if and only if it is T_1-odd and $T_1 \triangle T_2$-odd). A T_1-even, T_2-even cut is $\delta(S)$ for $S \subseteq V$ such that $|T_1 \cap S|$ is even and $|T_2 \cap S|$ is even. The uncrossing lemma does not follow in this case; in fact, it is an open problem to devise an efficient algorithm finding minimum T_1-even, T_2-even cuts.

5.2.2 Minimally Imperfect Graphs

A matrix $0, 1$ matrix M with no column of zeroes is *perfect* if the *set packing polytope* $\{x \in \mathbb{R}_+^n : Mx \leq \mathbb{1}\}$ is integral. One can show that a matrix is perfect if and only if it is the clique-node matrix of a perfect graph. Taking induced subgraphs corresponds to taking column submatrices and we define matrix M to be *minimally imperfect* if the corresponding set packing polytope is not integral but the set packing polytope for every column submatrix is integral. The minimally imperfect graphs are known to be the odd holes and odd antiholes. There exist polynomial time algorithms to recognize if a graph has an odd hole or odd antihole and thus check if M is perfect. For more background information about this problem see, [6] Chapter 3.2 and Chapter 3.3.

A result akin to Lehman's Theorem for minimally non ideal clutters holds for minimally imperfect graphs. Bruce Shepherd [31] asked if an analogue to the constructive version of the Lehman's Theorem (Theorem 3.1.3) holds for minimally imperfect graphs. This could lead to alternative algorithms for finding odd holes or antiholes. Since the corresponding separation problem requires checking for given k if there is a clique of size at least k and thus is hard to solve, no analogous result to Theorem 3.1.3 is possible.

5.3 Conclusions

In conclusion, we used a constructive version of Lehman's Theorem to show that for lifts of graphic and cographic matroids there exist polynomial time algorithms finding either a fractional flow and cover of the same value, a cut condition violation, or an obstruction. Our attempts to get analogous results for Eulerian integer multi-commodity flows were foiled, leaving this as an open problem.

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