2-crossing critical graphs with a V_8 minor

by

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AUTHOR'S DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Elizabeth Austin

Abstract

The crossing number of a graph is the minimum number of pairwise crossings of edges among all planar drawings of the graph. A graph G is k-crossing critical if it has crossing number k and any proper subgraph of G has a crossing number less than k.

The set of 1-crossing critical graphs is is determined by Kuratowski's Theorem to be $\{K_5, K_{3,3}\}$. Work has been done to approach the problem of classifying all 2-crossing critical graphs. The graph V_{2n} is a cycle on $2n$ vertices with n intersecting chords. The only remaining graphs to find in the classification of 2-crossing critical graphs are those that are 3-connected with a V_8 minor but no V_{10} minor.

This paper seeks to fill some of this gap by defining and completely describing a class of graphs called fully covered. In addition, we examine other ways in which graphs may be 2-crossing critical. This discussion classifies all known examples of 3-connected, 2-crossing critical graphs with a V_8 minor but no V_{10} minor.

Acknowledgements

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Chapter 1

Introduction

The *crossing number* cr(G) of a graph G is the minimum number of pairwise crossings of edges among all planar drawings of the graph. A graph G is k-*crossing critical* if cr(G) $\geq k$ and cr($G\backslash\{e\}$) $< k$ for every edge $e \in E(G)$. In fact, since vertices of degree 2 can have no effect on crossing number [3], we can consider only graphs with minimum degree at least 3. By Kuratowski's Theorem, we know that the only minimal 1-crossing critical graphs are K_5 and $K_{3,3}$.

There has been a great deal of work done on the problem of 2-crossing critical graphs. The first such graphs were discovered by Bloom, Kennedy and Quintas [1]. A family of graphs discovered by Šiřáň shows that there are infinitely many such graphs [9]. The only 2-crossing critical graph with crossing number greater than 2 is $C_3\square C_3$, the Cartesian product of two 3-cycles, which has crossing number three [15]. There is a complete characterization known for cubic graphs [12].

The characterization for graphs that are not 3-connected can be found in Section 14 of [3]. This characterization uses the easily seen fact that crossing number is additive over blocks. We form the 2-crossing critical graphs that are not 2-connected from two blocks, each of which is a subdivision of $K_{3,3}$ or K_5. Each block has at most one subdivided edge, and any new vertex must be the cut vertex of the graph. This gives 13 2-crossing critical graphs.

For the 2-crossing critical graphs that are 2-connected but not 3-connected, we need a different way to decompose the graphs. We use Tutte's decomposition of a 2-connected graph into cleavage units. It can be shown with some work that any such 2-crossing critical graph has at most three cleavage units,

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at most two of which are non-planar. If the graph has three cleavage units, then one of them must be a 3- or 4-cycle connecting the other two, which are non-planar. There are 16 2-crossing critical graphs with two cleavage units, and 20 with three cleavage units.

The only remaining case to considered is when there is just one non-planar cleavage unit C. Although not trivial, it can be shown that a 2-crossing critical graph G can be obtained from C by replacing some edges of C with digonal paths, that is, paths in which every edge is part of a parallel pair. If these edges are replaced with parallel pairs instead of digonal paths, then the resulting graph is both 3-connected and 2-crossing critical.

While Širáň's graphs used multiple edges to achieve criticality, Kochol defined a series of infinite families of simple graphs, $G(n, k)$, where for each $k \in \mathbf{N}$ $G(n, k)$ is n-crossing critical [6]. Since we are working with 2-crossing critical graphs, we will consider $G(2, k)$.

The graph $G(2, k)$ consists of $2k + 1$ cycles of length five on the vertices $5i + 1, 5i + 2, ..., 5i + 5$ for $0 \le i \le 2k$. The jth cycle is connected by three edges to each of the $j - 1$th and $j + 1$st cycle, modulo $2k + 1$, as shown in Figure 1.1. This is, in fact, a special case of a more general classification of 2-crossing critical graphs we will see in our discussion of tiles.
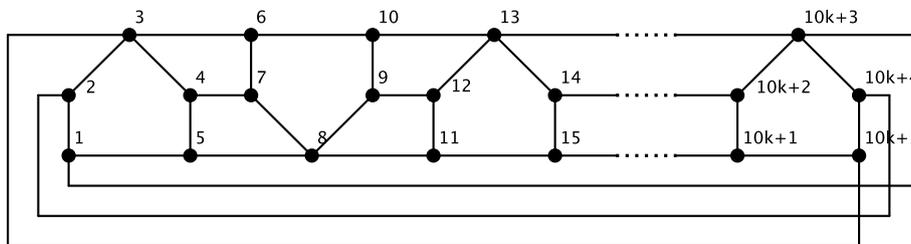
Figure 1.1: $G(2, k)$

There has been a lot of work on average degrees that can occur in infinite families of k-crossing critical graphs. The basic question can be formulated as follows. Let r be a rational number and let k be a positive integer. Determine whether or not there is there an infinite set of simple, minimum degree at least 3, k-crossing critical graphs so that each one has average degree r.

Salazar did the first work on this problem, showing the existence of infinite families of k-crossing critical graphs to satisfy $4 \le r < 6$, for infinitely many values of k [13]. Pinontoan and Richter expanded on this work by introducing

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tiles [11]. A *tile* is a triple $T = (G, L, R)$ such that G is a graph, and L and R are disjoint sequences of vertices of G. A *tile drawing* of a tile $T = (G, L, R)$ is a drawing of G in the unit square such that the vertices of L occur in order along the line $x = 0$ and the vertices of R occur in order along the line $y = 0$.

We can use such drawings to consider the crossing number of the tile. The *tile crossing number* of a tile T is the minimum number of pairwise crossings of edges among all planar drawings of the graph. If the tile crossing number is 0, then we say the tile is *planar*. A tile is *perfect* if: both $G - L$ and $G - R$ are connected; for each $v \in L$ (R), there is a path to R (L) disjoint from L (R) apart from v; for each $i \neq j$, there are disjoint paths from L_i to R_i and from L_j to R_j.

Two tiles, $S = (G, L, A)$ and $T = (H, B, R)$ where $A = (A_1, A_2, \ldots, A_{|A|})$ and $B = (B_1, B_2, \ldots, B_{|B|})$, are *compatible* if $|A| = |B|$ and, for each $1 \leq i, j \leq |A|$, $A_i A_j \in E(G)$ if and only if $B_i B_j \in E(H)$. To form the tile ST, we identify A_i with B_i for each $1 \leq i \leq |A|$.

If T is compatible with itself, then we form $\circ(T)$ by identifying B with R. T^n is defined inductively, as follows: $T^1 = T$, $T^n = T^{n-1}T$. The tile $\widetilde{T}$ is T with the sequence R reversed.

We define two tile combinations to help in finding crossing critical graphs with a set average degree. Let $T^{\bowtie}(m, n) = T^m \widetilde{T} T^n$ and $T^{\otimes}(n) = \circ(T^n \widetilde{T})$. For a perfect planar tile T, there are integers n, m and M such that for every $n \geq M$, $\mathrm{cr}(T^{\otimes}(n)) = \mathrm{tcr}(T^{\bowtie}(m, m))$. Moreover, $\mathrm{tcr}(T^{\bowtie}(m, m))$ is a non-increasing sequence and so it is eventually constant. This allows us to determine the crossing number of $T^{\otimes}(n)$ for a given tile.

Let $T_{h,s,m}$ be the tile formed by the tile $S^s B^m$ and h edges, the top one of which is identified with the path formed by the bottom of $S^s B^m$, as shown in Figure 1.3. The graph $T^{\otimes}_{h,s,m}(n)$ is $\binom{2h+3}{2}$-crossing critical when n is sufficiently large [11].
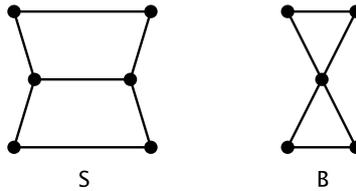

Figure 1.2: The tiles S and B
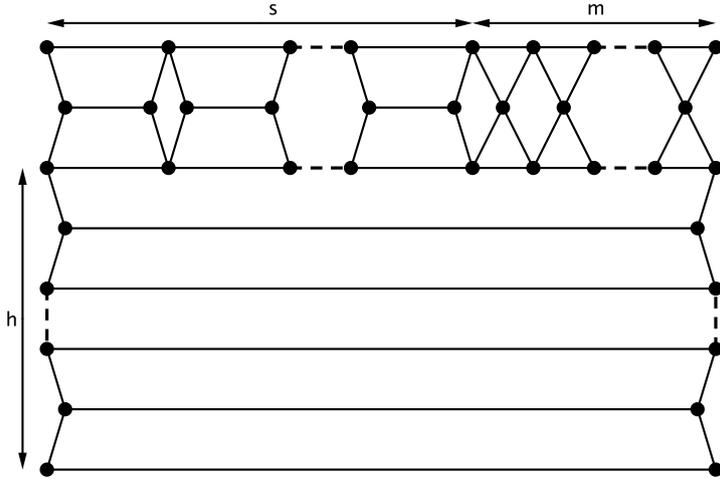
Figure 1.3: The tile $T_{h,s,m}$

Moreover, $T_{h,s,m}^{\otimes}(n)$ had average degree $\frac{14s+12h+12m}{4s+3h+3m}$. For any $r \in (3.5, 4)$ and any positive integer h, there are infinitely many choices of s and m such that $T_{h,s,m}^{\otimes}(n)$ has average degree r. This gives an infinite set of positive integers k for which we have an infinite family of k-crossing critical graphs with average degree r.

Bokal built from this work by introducing the zip product, which he combined with the work done on tiles to find examples for $3 < r < 6$ [2].

For the remaining classifications, we need to introduce a family of graphs. The graph V_{2n} consists of a cycle $(v_1, v_2, ..., v_{2n}, v_1)$ on $2n$ vertices with a chord between each pair of vertices v_i and v_j such that $|i - j| = n$. For each $k \geq 3$, it has been shown that all large 3-connected 2-crossing critical graphs contain a V_{2k} minor [5].

Let $\mathcal{S}$ be the set of tiles formed by placing one of the thirteen pictures in Figure 1.4, either in the orientation shown or rotated 180°, into one of the two frames in Figure 1.5.

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Figure 1.4: Pictures to form the tiles of $\mathcal{S}$

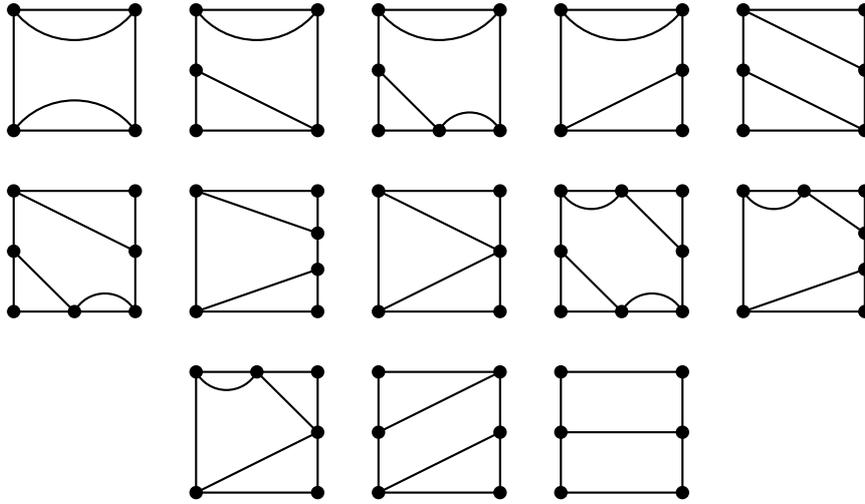Figure 1.5: Frames to form the tiles of $\mathcal{S}$

Let $\mathcal{T}$ be the sequence $(T_0, \widetilde{T}_1, T_2, \ldots, \widetilde{T}_{2m-1}, T_{2m})$, where each $T_i \in \mathcal{S}$ and $m \geq 1$. The set $\mathcal{T}(\mathcal{S})$ is the set of all graphs of the form $\otimes \mathcal{T}$. Each of the graphs in $\mathcal{T}(\mathcal{S})$ is 2-crossing critical, and all the V_{10}-containing, 3-connected, 2-crossing critical graphs are contained in this set [3]. Kochol's graph $G(2, k)$ as described above is also of this type, and can be constructed with the last picture and first tile.

The 2-crossing critical graphs with no V_8 minor are also known, and a finite number of graphs remain to be found [3]. These remaining graphs are those 3-connected, 2-crossing critical graphs with a V_8 minor but no V_{10} minor. This work was begun by Isabel Urrutia-Schroeder in her Master's essay [14] and will be continued here. Oporowski developed a list of 531 2-crossing critical graphs, 201 of which have a V_8 minor but no V_{10} [10]. Clearly, any complete characterization must include all of these graphs, and so this paper will seek to explain the characteristics that make these 201 graphs 2-crossing critical.

In Chapter 2, we examine the 1-drawings of a V_8. We define the term fully covered as a class of 2-crossing critical graphs. All possible single edges that can be added to a V_8 are considered and the effects they have on the possible 1-drawings of the resulting graph.

In Chapter 3, we will fully describe the 3-connected, 2-crossing critical graphs with a V_8 minor but no V_{10} minor that are fully covered. The first section of this corresponds to the work done by Urrutia-Schroeder, and corrects some inaccuracy in that work. The paper claims to find 326 non-isomorphic, 2-crossing critical graphs, but only 214 of those graphs were in fact 2-crossing critical. With this error fixed, we extend the work to include some larger structures not previously considered, giving a total of 312 graphs.

In Chapter 4, some other ways to achieve criticality in such 2-crossing graphs will be discussed in lesser depth. Of Oporowski's 201 graphs, all but 8 satisfy our definition of fully covered. These remaining graphs are critical for three reasons, each of which will be examined.

Chapter 2

Fully Covering a V_8

The graph V_8 is an 8-cycle with four chords joining the pairs of vertices at distance 4 on the cycle, as shown in Figure 2.1. We call the 8-cycle the *rim* and the four chords *spokes*. Each edge of the rim is a *rim branch*. As we add structures to the V_8, its edges may become subdivided. In this case, the spokes and rim branches may be paths rather than edges, but they retain the same designation. The (subdivisions of) 4-cycles created by consecutive spokes and the two rim branches between them are *quads*.
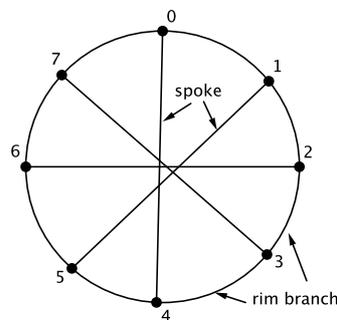

Figure 2.1: V_8

We will generally show graphs as drawn on the Möbius strip for ease of representation.
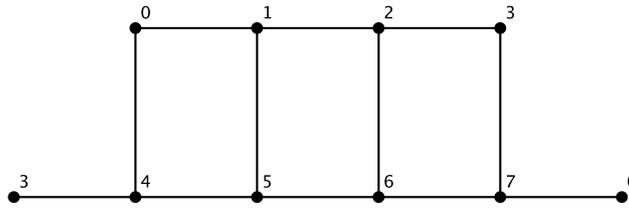
Figure 2.2: V_8 embedded on a Möbius strip

It is from this base that we build our 2-crossing critical graphs by subdividing and adding edges. We are concerned with extending a V_8 to a 2-crossing critical graph without creating a V_{10}.

To do this, we analyze 1-drawings of V_8. We define all possible single-edge additions and what possible crossings of the V_8 they prevent. We examine how these single-edge additions can interact to effect the possible crossings. In doing this, we will check Urrutia-Schroeder's work with small structures, comparing it to our findings with those structures.

2.1 Working with a V_8

It is easy to see that V_8 has crossing number 1. A drawing of a graph G with at most one pair of crossed edges is a 1-*drawing* of G. Since it has a $K_{3,3}$ minor, V_8 cannot be planar, but it does have 1-drawings, as shown in Figure 2.3. In this section, we will show that these two 1-drawings are the only non-isomorphic possible 1-drawings of V_8.
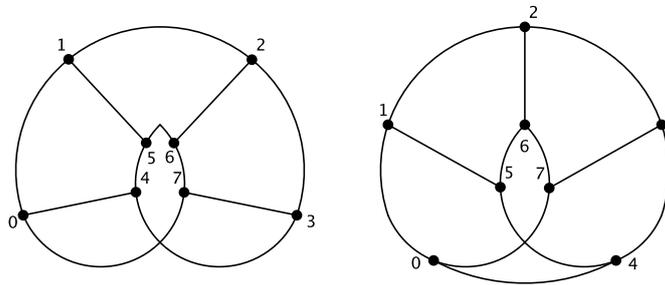

Figure 2.3: Two 1-drawings of V_8

Let us consider which pairs of edges in a graph can be crossed in a 1-drawing. To determine this, we have two basic tools.

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Lemma 2.1. *Disjoint cycles do not cross in a 1-drawing.*

PROOF Let C_α and C_β be disjoint cycles in a graph G. Any two disjoint cycles must cross an even number of times; for each time C_α crosses into C_β, it must also cross out. Since there are fewer than 2 crossings in any 1-drawing, the cycles must intersect exactly zero times. Therefore, no edge in C_α can cross any edge in C_β. □

Lemma 2.2. *Let α be an edge in a graph G. If $G\backslash\{\alpha\}$ has a $K_{3,3}$ minor, then α is not crossed in a 1-drawing of G.*

PROOF If α is crossed in a 1-drawing of G, then $G\backslash\{\alpha\}$ is planar and so contains no $K_{3,3}$ minor. □

With these two facts in mind, we can eliminate several pairs of edges from being crossed 1-drawings of V_8. The rim branch from i to $i+1$ is denoted r_i, where all values of i are taken to be modulo 8. The spoke from j to $j+4$ is denoted s_j, where all values of j are taken to be modulo 4. We indicate the inclusion or exclusion of endpoints by using square or angle brackets, respectively. For example, $[0, r_0, 1\rangle$ indicates the rim branch from 0 to 1, including 0 and excluding 1.

Lemma 2.3. *No spoke is crossed in a 1-drawing of a V_8.*

PROOF The graph $V_8\backslash s_j$ is a subdivision of $K_{3,3}$. Therefore, by Lemma 2.2, the spoke cannot be crossed. □

Lemma 2.4. *If two rim branches r_i and r_j are crossed in a 1-drawing of a V_8, then $|i - j| = 3$ or 4.*

PROOF Suppose two adjacent rim branches, r_i and r_{i+1}, are crossed in a 1-drawing of a V_8. Removing the spoke s_{i+1} leaves a 1-drawing of $K_{3,3}$ in which the single crossing is one edge crossing itself. Since $K_{3,3}$ has crossing number 1, there must be a crossing between two distinct edges, so such a drawing is not possible. Thus, it is not possible to have a 1-drawing of the V_8 in which adjacent rim branches are crossed.

Two rim branches r_i and r_{i+2} are on disjoint quads. By Lemma 2.1, they cannot be crossed in a 1-drawing.

Therefore any crossed rim branches must be at distance 3 or 4, as required.

$\square$

Figure 2.3 shows the two non-isomorphic embeddings of V_8: the left hand one has r_i crossing r_{i+4}, while the right hand one has r_i crossing r_{i+3}. The two embeddings are necessarily non-isomorphic, as the crossing in one is incident with faces having 2, 2, 4 and 4 vertices, while the other's crossing is incident with faces having 2, 3, 3 and 3 vertices . We call the pairs of edges in opposite or next to opposite rim branches *crossing pairs*, as they are the pairs of edges that can be crossed in a 1-drawing.

2.2   Structures and Covering

In order to create 2-crossing critical graphs, we will add structures to the V_8. For our purposes, these structures are single edges with each endpoint on the V_8, possibly subdividing one of its edges. Each structure added can prevent certain pairs of edges from crossing in a 1-drawing. We say that an edge on the rim is *covered* if each crossing pair involving that edge is prevented in a 1-drawing by one or more structures.

A V_8 is *fully covered* if all crossing pairs are eliminated. It is easy to see that a fully covered V_8 has crossing number at least 2, since there is no pair of edges remaining whose crossing can yield a 1-drawing. Our goal in this work is to find all 2-crossing critical graphs that have a fully covered V_8. In this section, we consider all possible single-edge additions to a V_8 and what effect they have on the possible crossings of the V_8 in a 1-drawing of V_8 plus the addition. The following observation limits the amount of checking we need to do.

Lemma 2.5. *If five consecutive rim branches of a V_8 in a graph G are covered, then G is fully covered and has crossing number at least two.*

Proof With five rim branches covered, any possible crossing pair must consist of edges from the remaining three. These rim branches are consecutive and so are at a distance of at most 2 from each other. But, by Lemma 2.4, rim branches can only cross in a 1-drawing if they are at a distance of 3 or 4. Then that V_8 is fully covered. In particular, G has crossing number at least 2.

$\square$

What happens when we add a single edge to a V_8? Depending on where the edge is added, it will cover different sections of the rim. Note that it is not possible for the new edge to be crossed in a 1-drawing, since any drawing must have at least one crossing in the original V_8 by virtue of V_8 having crossing number 1.

We divide the new structures added to the V_8 into three categories: jumps, slopes and bars. All possibilities are defined and pictured below. The dotted lines in the figures show the sections of the rim covered by the added structure. We define the *span* of a structure to be the section of rim between the endpoints of that structure.

A *jump* is an edge with both endpoints on the rim of the V_8. For $k = 1, 2$, a k-jump has endpoints i and $i + k$, spanning k rim branches. For $k = 0, 1, 2$, a $k\frac{1}{2}$-jump has one endpoint at i and the other on r_{i+k} or r_{i-k-1}, spanning k full branches and part of another. For $k = \frac{1}{2}, 1, 2, 3$ an off k-jump has endpoints on r_i and $r_{\lfloor i+k \rfloor}$, spanning k full rim branches and parts of two others or, in the case of the off $\frac{1}{2}$-jump, spanning part of a single rim branch.
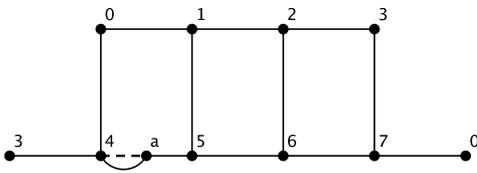

Figure 2.4: $\frac{1}{2}$-jump
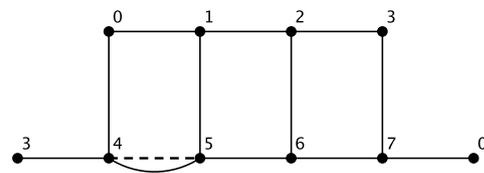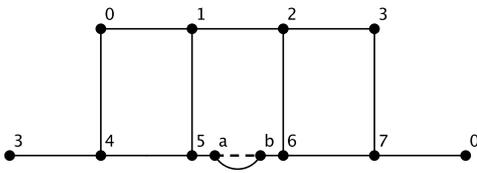

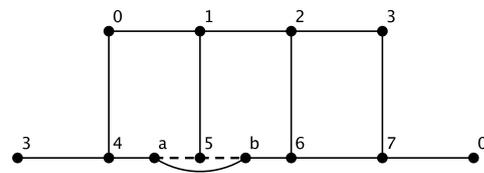Figure 2.5: 1-jump

Figure 2.6: Off $\frac{1}{2}$-jump

Figure 2.7: Off 1-jump

Jumps can also be placed on the spokes of the V_8, in which case they are denoted *spoke jumps*. These do not eliminate any crossings.
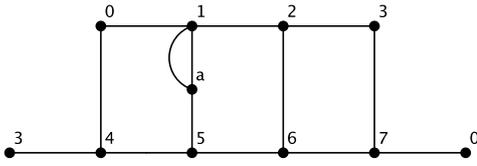
Figure 2.8: $\frac{1}{2}$-spoke jump
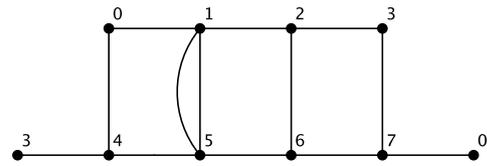
Figure 2.9: Spoke jump
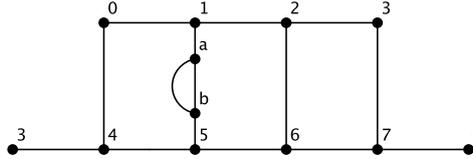
Figure 2.10: Off spoke jump

A *slope* is an edge with one endpoint on the rim and the other on a spoke. For $k = 1, 2$, a k-slope has one endpoint on s_i and the other at $i + k$ or $i - k$, spanning k rim branches. For $k = 0, 1$, a $k\frac{1}{2}$-slope has one endpoint on s_i and the other on r_{i+k} or r_{i-k-1}, spanning k rim branches and part of another.
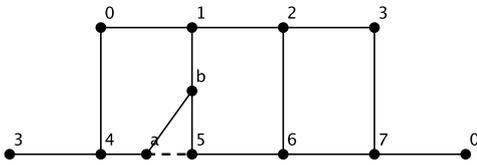
Figure 2.11: $\frac{1}{2}$-slope
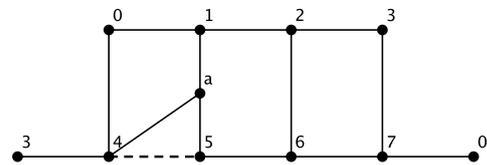
Figure 2.12: 1-slope

An edge between two spokes is a *bar*. A bar has endpoints on s_i and s_{i+1} and spans the two rim branches between these spokes. A 2-bar has endpoints on s_i and s_{i+2} and, since this can also be considered to be a bar from s_{i+2} to s_i, spans the entire rim.
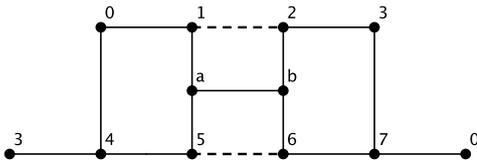
Figure 2.13: Bar

Figure 2.14: 2-bar

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It is more convenient to look at some of the larger jumps in a slightly different way. A 3-jump can be drawn as a chord in a quad and is denoted a *diagonal*. A $3\frac{1}{2}$-jump is also an edge from a vertex i across a quad to r_{i+4}, called a *semi-diagonal*. The diagonal alone place no limits on crossings, and the semi-diagonal only eliminates one crossing.
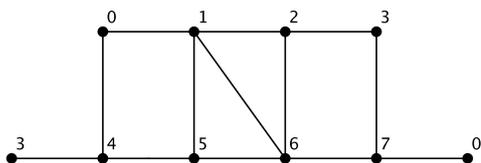
Figure 2.15: Diagonal
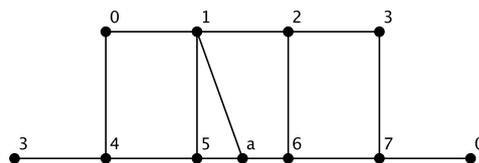
Figure 2.16: Semi-diagonal

There are only a few other possible edges that could be added to the V_8. A jump of length 4 is a spoke jump. Any larger jumps are eq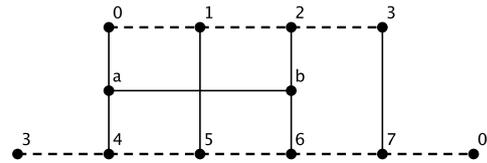uivalent to smaller ones. The only off jumps not given here are the off 4-jump, which would add a fifth spoke and make a V_{10}, and the off 2-jump, which we will show is equivalent to the bar. Therefore, the 19 structures discussed here are the only possible edges to add to the V_8 in our context.

In order to find fully-covered graphs, we need to understand what coverage is given by each of the structures.

Theorem 2.6. *Consider a graph consisting of a V_8 with some structure S added.*

1. *(a) If S is a slope, a bar or a k-jump, with $k \leq 2$, then it covers the section of rim it spans.*

   (b) If S is a $2\frac{1}{2}$-jump or off 3-jump, then it covers the two full rim branches it spans.

2. *(a) If S is a $1\frac{1}{2}$-slope or 2-slope from s_i that spans r_i, then it also covers r_{i+2}, r_{i+3}, r_{i+5} and r_{i+6}.*

   (b) If S is a $2\frac{1}{2}$-jump or off 3-jump from r_i that spans r_{i+1}, then it also covers r_{i+5} and, in the case of an off 3-jump, r_{i+6}.

3. *(a) If S is a $2\frac{1}{2}$-jump from r_i to $i+3$, then the section of r_i spanned by S can only cross r_{i+3}. If S is an off 3-jump from r_i to r_{i+3}, then the section of r_i spanned by S can only cross r_{i+3} and the section of r_{i+3} spanned by S can only cross r_i.*

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(b) If S is a semi-diagonal from i to r_{i+4}, then the section of r_{i+4} spanned by S cannot cross r_{i+1}.

PROOF

1. If S is a jump or slope, delete the edges that we claim are covered by S. If S is a bar from s_i from s_j, then delete the rim from r_i to r_{j-1}. If deleting these edges leaves one of the vertices k with degree 1, then delete s_k. Otherwise delete any spoke. In each case, we delete one spoke, leaving a revised rim with three spokes; this is a subdivision of $K_{3,3}$, which has crossing number 1. By Lemma 2.2, none of the deleted edges can be crossed in a 1-drawing of the graph. The remaining coverage caused by the bars follows by symmetry.

2. (a) Let S be a slope with endpoints a and b on s_i and r_{i+1} or $i+1$. Define α to be the cycle $(i+4, a, b, i+2, i+3, i+4)$ and β to be $(i, i+1, i+5, i+6, i+7, i)$. By Lemma 2.1, these disjoint cycles guarantee the required coverage.
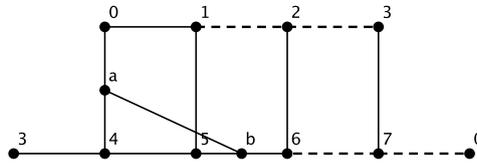
Figure 2.17: Extra coverage by $1\frac{1}{2}$-slope (2(a) with $i = 4$)
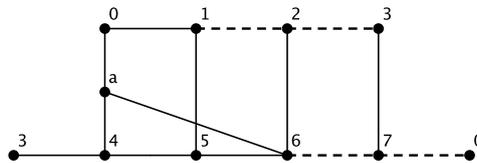
Figure 2.18: Extra coverage by 2-slope (2(a) with $i = 4$)

(b) If S is a $2\frac{1}{2}$-jump or off 3-jump, then removing the specified edges as well as either r_{i+2} if S is a $2\frac{1}{2}$-jump or s_{i+2} if it is an off 3-jump, leaves a subdivided $K_{3,3}$.

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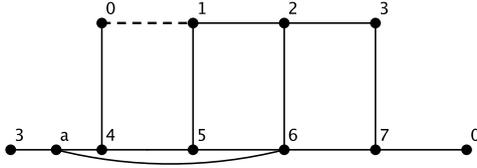
Figure 2.19: Extra coverage by $2\frac{1}{2}$-jump (2(b) with $i = 3$)
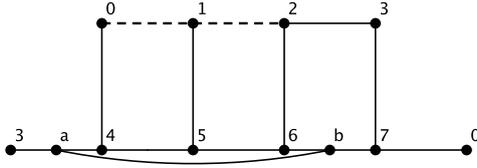
Figure 2.20: Extra coverage by off 3-jump (2(b) with $i = 3$)

3. (a) Let α be the cycle formed by S and the section of rim it spans. Let β be the cycle $(i + 5, i + 6, i + 7, i, i + 4, i + 5)$. Then the cycles are disjoint and Lemma 2.1 eliminates all crossing pairs involving the partial rim branch or rim branches spanned by S except those specified.

   (b) Denote the endpoint of S on r_{i+4} by v. Let α be the cycle $(i, i + 4, v, i)$ and β be the cycle $(i+1, i+2, i+6, i+5, i+1)$. By Lemma 2.1, the rim from $i + 4$ to v cannot cross r_{i+1}.

   $\square$

Clearly we can eliminate all spoke jumps and the diagonal from consideration, since they do not eliminate any crossing pairs. This does not necessarily prevent them from being in a 2-crossing critical graph (Figure 4.4, for example, shows a 2-crossing critical graph with a diagonal), but the criticality would have to achieved in some way other than full coverage as we have defined it here.

Lemma 2.7. *If G is a fully covered, 2-crossing critical V_8 with an off 2-jump, then G is also a fully covered, 2-crossing critical V_8 with no off 2-jump.*

PROOF The transformation shown in Figure 2.21 eliminates an off 2-jump by choosing a new V_8 with a bar. We must show that this transformation maintains coverage, and that it does not create any off 2-jumps.

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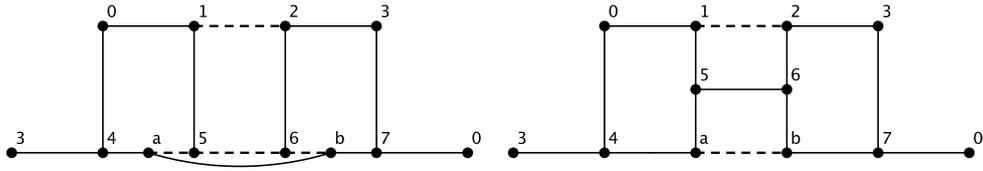
Figure 2.21: Transformation from an off 2-jump to a bar

Suppose there is a structure S with an endpoint at a. Then the length of S will be increased by $\frac{1}{2}$ after the transformation and so, checking the cases, it is easy to see that the new structure prevents the same crossings unless it is a $2\frac{1}{2}$-jump, off 3-jump or $1\frac{1}{2}$-slope to a spoke spanned by the off 2-jump. A simple check shows that the off 3-jump gives a graph with crossing number 2 that is not critical. The other two are 2-crossing critical graphs in which we can find a fully covered V_8 with no off 2-jump, but an off 3-jump instead. By symmetry, the same is true with an endpoint on b.

If S has an endpoint on $\langle a, r_4, 5]$, then that point will be on a spoke after the transformation. After the transformation, S will prevent at least those crossings that were prevented before unless its endpoint is on $\langle a, r_4, 5 \rangle$ and it is a $2\frac{1}{2}$-jump or off 3-jump, or a $1\frac{1}{2}$-slope to a spoke within the off 2-slope. However, by using the transformation in Figure 2.22, each of these can be treated as a fully-covered V_8 with no off 2-jump but an off 3-jump instead. This also covers $[6, r_6, b\rangle$, by symmetry.
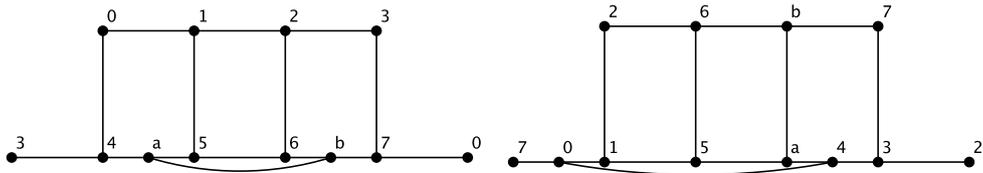
Figure 2.22: Transformation from off 2-jump to off 3-jump with $i = 4$

If S has an endpoint on r_5, then the graph cannot be fully covered and 2-crossing critical. Any other structures are unchanged by the transformation and therefore eliminate the same crossings as in the original drawing. $\square$

The transformation in Figure 2.22 shows that the off 3-jump eliminates the same crossing pairs as the off 2-slope or bar, so it is conceivable that we could eliminate the off 3-jump from consideration as well. However, using the off 3-jump offers possibilities not given by the bar. The sections of rim we cover

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in order to get five in a row are equivalent for the two drawings. Covering from b to 1 with the off 3-jump gives five in a row with the bar covering the first and last. The only possible coverage of these rim branches with small structures that does not translate to a covering using only bars and jumps of length 2 or less is using a 2-jump. When we move to the bar representation, this becomes a 3-jump, or diagonal, which generally does not give any coverage. In this way, we can find more fully covered graphs with the off 3-jump than we can with the bar alone.

Chapter 3

Finding Fully Covered Graphs

Knowing the coverage provided by each structure, we can begin to place multiple structures on a V_8 in order to increase the crossing number of the resulting graph to 2. Our goal is to find all combinations of these structures that yield 2-crossing critical graphs.

In order to facilitate this search, we build the graphs in three stages, described individually in the first three sections of this chapter. Stage 1 deals with the simplest of the structures – those that cover only the section of the rim they span. Stages 2 and 3 add the larger jumps and slopes, respectively. In the final section, we discuss the algorithm used to build these graphs.

3.1   Stage 1 – small and simple

In this section we treat the case of adding only structures that cover exactly the section of rim they span. This contains the work by Urrutia-Schroeder. These structures, which we will refer to as *small structures*, are as follows:

- $\frac{1}{2}$-jump

- 1-jump

- $1\frac{1}{2}$-jump

- 2-jump

- off $\frac{1}{2}$-jump

- off 1-jump

- $\frac{1}{2}$-slope

- 1-slope

- bar

- 2-bar

We want to put these structures on a V_8 in such a way that five consecutive

rim branches are covered; by Lemma 2.5, this is equivalent to having a fully covered graph. We will do this by placing structures on the rim branches from 0 to 5 in all possible combinations. This will always guarantee a crossing number of at least two, but we also need to determine which combinations yield 2-crossing critical graphs.

Two structures S and S' are *disjoint* if the sections of the rim covered by S and S' have no edge in common. The coverage considered is that provided by Theorem 2.6.

Theorem 3.1. *When covering* 5 *sequential rim branches with any combination of bars, 2-bars and $\frac{1}{2}$-, 1, $1\frac{1}{2}$, 2-, off $\frac{1}{2}$ and off 1-jumps, the sections of rim covered by the structures must be disjoint in order to have a 2-crossing critical graph.*

PROOF First, we note that by choosing a different V_8, H', which is the same as the original, H, except for the edges defining a single rim branch, we do not affect the other rim branches. That is, we can redefine the rim in one area without changing which sections of rim are covered by disjoint structures.

Evidently, if any structure S covers only sections of the rim already covered by other structures, then S can be removed without affecting the coverage. For example, if a V_8 has a 2-bar, then, since the 2-bar covers the entire rim, the coverage from any other structure would be redundant, and so does not exist. Likewise, we cannot have an off 1-jump whose span is contained in the span of a bar or $\frac{1}{2}$-, 1-, $1\frac{1}{2}$-, 2 or off 1-jump. The other instances of such an overlap are a 1-jump occuring in the span of a bar, $1\frac{1}{2}$-jump or 2-jump, an off 1-jump in the span of a $1\frac{1}{2}$-jump or 2-jump, or a $1\frac{1}{2}$-jump in the span of a 2-jump. Since we are looking for critical graphs, we can discount any such situation.

The remaining possibilities are outlined in Table 3.1, where J and S are the overlapping structures, with the number of different ways they can overlap indicated. S' is the structure found by redefining the rim of the V_8 to go through J and removing an edge. Given any pair of structures, if possible we choose J to be as early as possible in the following list: off $\frac{1}{2}$-jump, $\frac{1}{2}$-jump, 1-jump, off 1-jump.

Case 1: J is an off $\frac{1}{2}$-jump, a $\frac{1}{2}$-jump or a 1-jump with endpoints a and b, a structure S has an endpoint c in the span of J. Assume, without loss of generality, that a is in the span of S. Then we can redefine the rim to go through J. Deleting $\langle a, c \rangle$, the area of the rim that was covered by S and

J	S	# possible	S'
off $\frac{1}{2}$-jump	off $\frac{1}{2}$-jump	1	off $\frac{1}{2}$-jump
off $\frac{1}{2}$-jump	$\frac{1}{2}$-jump	1	$\frac{1}{2}$-jump
off $\frac{1}{2}$-jump	off 1-jump	1	off 1-jump
off $\frac{1}{2}$-jump	$1\frac{1}{2}$-jump	1	$1\frac{1}{2}$-jump
$\frac{1}{2}$-jump	$\frac{1}{2}$-jump	1	1-jump
$\frac{1}{2}$-jump	$1\frac{1}{2}$-jump	2	$1\frac{1}{2}$-jump or 2-jump
$\frac{1}{2}$-jump	off 1-jump	2	off 1-jump or $1\frac{1}{2}$-jump
1-jump	$1\frac{1}{2}$-jump	1	2-jump
1-jump	off 1-jump	1	$1\frac{1}{2}$-jump
off 1-jump	off 1-jump	1	$1\frac{1}{2}$-slope
off 1-jump	$1\frac{1}{2}$-jump	2	$1\frac{1}{2}$-slope or 2-slope
off 1-jump	2-jump	1	2-slope
off 1-jump	bar	1	bar
$1\frac{1}{2}$-jump	$1\frac{1}{2}$-jump	3	$1\frac{1}{2}$-slope or 2-slope or 2cc
$1\frac{1}{2}$-jump	2-jump	2	2-slope or 2cc
$1\frac{1}{2}$-jump	bar	2	$2\frac{1}{2}$-jump or bar
2-jump	2-jump	1	2cc
2-jump	bar	1	$2\frac{1}{2}$-jump

Table 3.1: Possible overlaps and results

J is now covered by the single structure formed by S extended with $\langle b, c \rangle$. Given that the only possibilities for S to overlap such a J without completely covering it have length at most $1\frac{1}{2}$, the extension of S is at most a 2 jump, ensuring coverage. Since we can have the same coverage with an edge deleted, the graph cannot be critical.
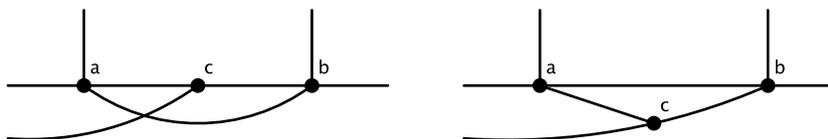
Figure 3.1: Alternate drawing of a 1-jump

Case 2: J is an off 1-jump with endpoints a in r_i and b in r_{i+1} and a structure S, either another off 1-jump, a $1\frac{1}{2}$-jump, a 2-jump or a bar, has an endpoint c in the span of J. Assume, without loss of generality, that a is in the span of S. Then we can redefine the rim to go through J, extending the spoke to include $\langle s, b \rangle$ and deleting the edge $\{a, c\}$. If the structure S is a jump, this turns it into a $1\frac{1}{2}$-slope or 2-slope, thereby covering a large portion of the rim, including the area previous covered by S and J. If S is a bar, we redefine the rim in the same way with the same edge deletion, leaving a bar that covers the same area.
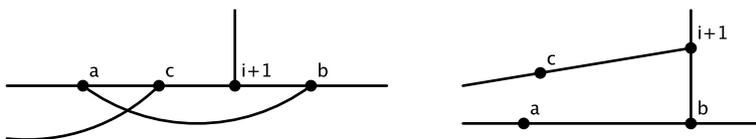
Figure 3.2: Alternate drawing of an off 1-jump

Case 3: Each of J and S is either a 2-jump or a $1\frac{1}{2}$-jump in which the partially covered rim branch is overlapped. When two 2-jumps overlap we can remove the edge covered by both, resulting in the 2-crossing critical graph $\mathcal{G}$, shown in Figure 3.3. The subgraph $\mathcal{G}$ can be found in a similar way when the rim branch partially covered by a $1\frac{1}{2}$-jump is overlapped by a $1\frac{1}{2}$-jump or 2-jump.
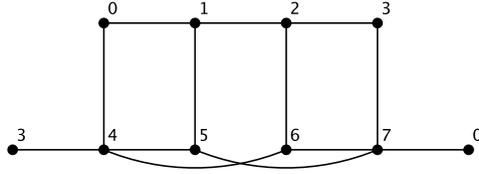
Figure 3.3: The graph $\mathcal{G}$

Case 4: J is a $1\frac{1}{2}$-jump from i to a point a in r_{i+1}, S is a $1\frac{1}{2}$-jump, 2-jump or bar with an endpoint x in the span of J, such that i is in the span of S. We can redefine the rim to go through J and remove the section of rim from i to either x if S is a jump, or $i+1$ if S is a bar. This creates a bar if we started with a bar, or a slope otherwise. The length of the slope will be $1\frac{1}{2}$ or 2, depending on whether the second endpoint of S is a main vertex of the V_8 or on a rim branch.

Case 5: J is a $1\frac{1}{2}$-jump or 2-jump from i to a vertex a on either r_{i+1} or $i+2$, S is a bar from s_{i+1} to s_{i+2}. We can redefine the rim to go through the bar and eliminate $\langle i+1, r_{i+1}, a \rangle$ in order to create a $2\frac{1}{2}$-jump that covers the same sections of rim as the previous jump and bar combined. $\qquad\square$

We need not include the $\frac{1}{2}$-slope or 1-slope when counting the combinations of small structures because we can transform them into an off 1-jump and $1\frac{1}{2}$-jump, respectively, using the transformations shown in Figures 3.4 and 3.5. In order to eliminate them, we must ensure that the transformations do not affect the coverage of the other rim branches. We will assume, in determining this, that we begin with a 2-crossing critical graph, as otherwise the graph is not relevant to the discussion.
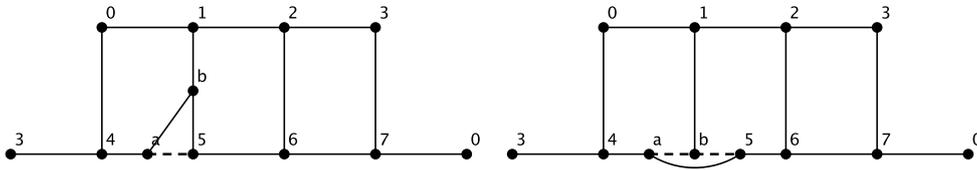

Figure 3.4: Transformation from a $\frac{1}{2}$-slope to an off 1-jump

The only structures that will be affected by the transformations are those with endpoints on $\langle a, r_4, 5, s_1, b]$ for the $\frac{1}{2}$-slope and on $\langle r_4, 5, s_1, a]$ for the 1-slope. Let J be the slope in question and S be a structure affected by the

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transformation, having endpoints u and v. Without any loss of generality, we take J to be positioned as shown in the figure.
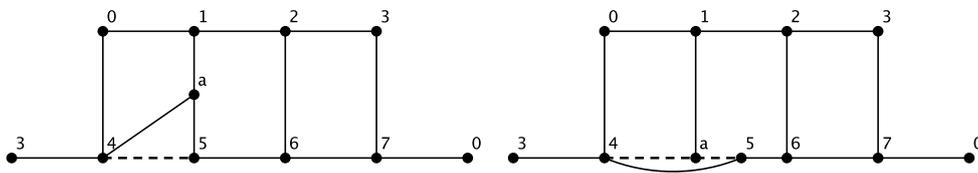

Figure 3.5: Transformation from a 1-slope to a $1\frac{1}{2}$-jump

Case 1: u is on $\langle b, s_1, 5 \rangle$. Then S must be a $\frac{1}{2}$-slope, 1-slope or bar, with v on $\langle 5, r_5, 6, s_2, 2 \rangle$. Using the transformation, the slopes become small jumps, covering the same edges. If S is a bar, then it becomes a 1-slope or $\frac{1}{2}$-slope. Repeating the transformation on this new slope gives two overlapping small jumps, which cannot happen in a critical graph by Theorem 3.1.

Case 2: u is 5. Then S must be either a jump with v on $\langle 5, r_5, 6, r_6, 7]$ or a slope with v on s_2. If S is a jump, the transformation makes a shorter jump that covers the same edges. Otherwise, S must be a 1-slope and therefore transforms into a $\frac{1}{2}$-slope. This leaves one fewer slope than the original graph, so a repetition of the transformations can still eliminate all of them.

Case 3: u is in the span of J. The only such possibility with S a slope is when both S and J are $\frac{1}{2}$-slopes. In that case, deleting the section of rim that was covered by both slopes in the original graph turns the two slopes into a $1\frac{1}{2}$-slope which covers even more than the two slopes separately. When S is a jump, deleting the doubly covered section of rim and transforming gives a larger jump. This new jump covers the same area as the two structures, unless S was a $1\frac{1}{2}$-jump from 3. That combination, however, gives a 2-crossing critical graph with no V_8 after deleting s_0, and so cannot occur in a critical graph in our context.

The remainder of this section is devoted to further understanding restrictions on how the structures may combine in a 2-crossing critical example.

A 2-bar on a V_8 gives the Petersen graph, which is 2-crossing critical. In fact, the 2-bar spans the entire rim of the V_8, therefore no other structure can be on the V_8 with a 2-bar. In this way, the Petersen graph is a fully covered graph.

Since we only consider 3-connected graphs and no two structures overlap, there can be at most one $\frac{1}{2}$-jump on any rim branch. If there was a rim

branch r_i with two $\frac{1}{2}$-jumps, then removing i and $i+1$ would disconnect the graph. Moreover, the only time there can be an off $\frac{1}{2}$-jump is when each of its endpoints is adjacent to a point outside of the rim branch containing the jump.

When placing the remaining structures, consider what happens when a structure has one endpoint in the five rim branches to be covered and the other endpoint outside that range. The only structures that can do this are the $1\frac{1}{2}$-jump, 2-jump and off 1-jump. Assume, without loss of generality, that J spans 0 and the five covered rim branches are r_0 through r_4. If the structure J going off the end has endpoints a on $[7, r_7, 0\rangle$ and b on $\langle 0, r_0, 1\rangle$, as is the case in Figure 3.6, then we can redefine the rim to go through that structure. Removing $\langle a, r_7, 0\rangle$ and redefining the spoke to be $\langle b, r_0, 0, s_0, 4\rangle$ leaves a fully covered graph. Thus the graph is not 2-crossing critical.
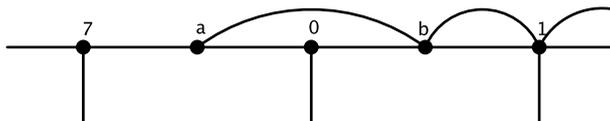

Figure 3.6: A structure going off the end of the 5 covered rim branches

A $1\frac{1}{2}$-jump from a vertex within the covered range to a rim branch outside the range covers a full rim branch and maintains criticality.

A 2-jump with one endpoint inside the covered range and the other outside, say from 7 to 1, covers a full rim branch outside of the range we need covered, from 0 to 5. So, in any critical graph with such a jump, we must be unable to remove the structures covering r_4 without also uncovering r_3. This means there must also be a 2-jump from 3 to 5. Moreover, if there is a bar between s_2 and s_3, then we have five consecutive rim branches covered from 6 to 3. This allows us to remove any structures from the rim branches from 3 to 5, meaning that the graph cannot be critical. Therefore, we can have a 2-jump going off the end of the designated five rim branches, say from 7 to 1, only if there is a 2-jump from 3 to 5 and no bar.

There are also restrictions on where we can place bars. Suppose there is a bar on the second of the five sequential covered rim branches. Then the rim branch directly following the five is also covered. This means that we need not

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cover the first of the five to have five in a row. We can, therefore, remove any structure covering that first rim branch. Assuming no overlap with the bar, any structure removed in this manner would not affect the coverage of the other four sequential rim branches unless it was a bar. We can, therefore, eliminate the bar from being on the second or, by symmetry, fourth rim branches of the five in a row unless there is also a bar on the first or, symmetrically, fifth.

When we do have bars on sequential quads in this way, they cannot share an endpoint. As shown in Figure 3.7, we can eliminate the undivided spoke and choose a new V_8 so that this configuration of bars turns into a 2-slope. If we cover the rim from 6 to 0 or from 2 to 4 in order to have five sequential covered rim branches using the bars, then we also form a 2-crossing graph in the version with the 2-slope, but with one of the original edges removed. Therefore the configuration in the first graph cannot yield a 2-crossing critical graph.
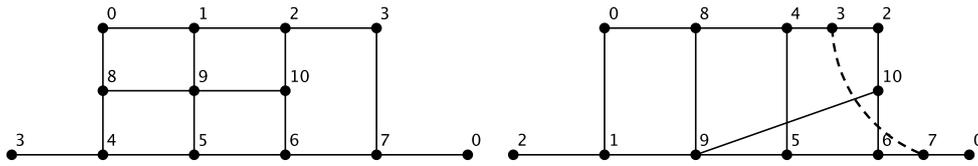
Figure 3.7: Transforming two bars to a 2-slope

Furthermore, if we have bars on three consecutive quads, then the graph has a V_{10} minor, as shown in Figure 3.8, and so does not fall into the category of graphs we are considering. Other arrangements of bars will give different slopes, but the same V_{10} minor.
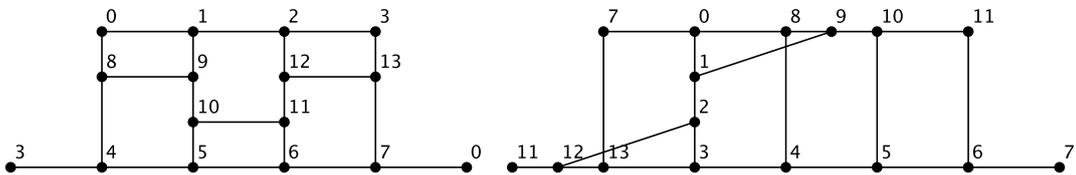
Figure 3.8: Three bars

So we can find all the 2-crossing critical graphs that use only these structures by examining all combinations of five of the eight rim branch configurations in Figure 3.9, as well as the 2-bar.

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Figure 3.9: The possible formations

The algorithm in Section 3.4 found exactly 231 non-isomorphic 2-crossing critical graphs consisting of a V_8 fully covered with these structures. This should be almost equivalent to the work done in [14], since there the search was limited to exactly the structures we have specified for this stage with the exception of the 2-bar. After accounting for the non-critical graphs counted in [14] and the one extra graph foun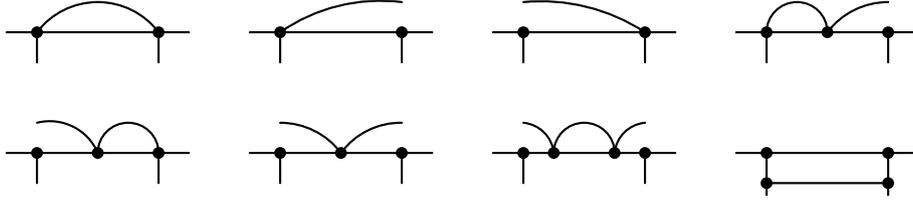d here with a 2-bar, there remain 16 new graphs. These 16 graphs all involve a structure going off the end of the five consecutive covered rim branches. The previous paper does not allow for 2-jumps to go off the end, and only allows for a $1\frac{1}{2}$-jump off of one end at a time.

3.2   Stage 2 – large jumps

In this section we extend the discussion to include the possibility of including $2\frac{1}{2}$-jumps and off 3-jumps, as well as the short structures of the preceding section. In doing this, our algorithm finds all fully covered V_8's with at least one large jump. These large jumps, shown in Figure 3.10, cover a section on the opposite side of the rim from where they are placed, as discussed in Theorem 2.6. For the $2\frac{1}{2}$-jump ab from a in r_3 to $b = 6$, the only ways to achieve full coverage are by covering the rim from 6 to 0, from 1 to 4 or from 2 to a and 6 to 7. For the off 3-jump ab with a in r_3 and b in r_6, covering either from 6 to 0 or from 2 to 4 are the only ways to fully cover the V_8. We may cover from 3 to 4 and b to 0 in lieu of 6 to 0, or 6 to 7 and 2 to a in lieu of 2 to 4.
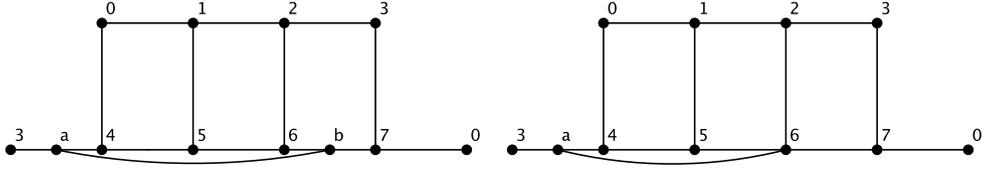
Figure 3.10: Off 3-jump and $2\frac{1}{2}$-jump

First let us consider covering the remaining sections with small structures, and the possible overlap between a small structure and a large jump. Any time one of these large jumps overlaps with a 2-jump, the resulting graph has crossing number two. If the large jump ab specified above is an off 3-jump and the 2-jump covers from 2 to 4 or 6 to 0, then the V_8 is also fully covered. However if ab is the $2\frac{1}{2}$-jump, then ab does not overlap a 2-jump; otherwise the two structures create a 2-crossing graph as will be discussed in Chapter 4. Similarly, we cannot have a fully covered V_8 in which a $1\frac{1}{2}$-jump or off 1-jump covers the endpoint of a $2\frac{1}{2}$-jump or off 3-jump.

If there is a structure S from s_3 to s_0 or $\langle a, r_3, 4]$ (so S is a bar, or a $\frac{1}{2}$-slope or a 1-slope), then we can choose a new V_8, using the large jump as a spoke and turning S into a $1\frac{1}{2}$-slope or 2-slope. Since the large slope covers so much more of the rim, we can eliminate at least one of the other structures from original graph and still have a fully covered V_8. This means that the original graph could not have been 2-crossing critical. By symmetry, the same is true for a structure from s_3 to s_2 or $[6, r_6, b\rangle$ with an off 3-jump.

If we have a fully-covered V_8 with a structure from s_0 to $[3, r_3, a\rangle$ in addition to the off 3-jump or $2\frac{1}{2}$-jump, then removing r_1 leaves a graph with no V_8 minor that has crossing number 2. Thus, this combination of structures cannot occur in a critical graph. If we have a jump from 3 to a vertex x on $\langle a, r_3, 4]$ and a bar from s_2 to s_3, then removing the rim from a to x leaves a graph with crossing number 2.

Suppose there is a structure S with endpoints x and y where x is on $\langle a, r_3, 4\rangle$ and y is on $\langle 4, r_4, 5]$. Then we can redefine the rim to go through S, with a spoke from 0, through 4, to x, and delete $\langle 4, r_4, y\rangle$. This leaves the large jump intact and maintains the same coverage as the original graph.

If $x = a$, we do the same process, but this shortens the large jump. The $2\frac{1}{2}$-jump turns into a 2-jump. However the only time $\langle a, r_3, 4\rangle$ would be covered is if all the rim from 1 to 4 is covered, so we do not need the extra coverage on

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r_0 originally provided by the large jump to have five consecutive rim branches covered. The off 3-jump turns into a $2\frac{1}{2}$-jump, so r_1 is still covered, but r_0 is not. There are two situations in which $\langle a, r_3, 4 \rangle$ may be covered. If we are covering the rim from 2 to 4, then there is still coverage from 1 to 6, giving the five rim branches needed. Otherwise $\langle a, r_3, 4]$ was covered in order to eliminate the crossing pair with $\langle 6, r_6, b \rangle$. In this case, $\langle 3, r_3, a \rangle$ must also be covered and it provides the covering needed to have the rim covered from 3 to 0, again giving five rim branches.

If x is on $\langle 3, r_3, a \rangle$ and y on r_4, then we redefine the rim to go through S with a spoke from 0 through 4 and a to x, and delete $\langle 4, r_4, y \rangle$. This turns the large jumps into slopes: the $2\frac{1}{2}$-jump into a 2-slope and the off 3-jump into a $1\frac{1}{2}$-slope. In any situation requiring $\langle a, r_3, 4 \rangle$ to be covered, $\langle 3, r_3, a \rangle$ would be also be covered, and that section of the rim together with the newly formed slope provides enough coverage to force a second crossing.

If we have a $2\frac{1}{2}$-jump with r_6 being covered by smaller structures, then any overlap onto the rim covered by the $2\frac{1}{2}$-jump will result in a variant of $\mathcal{G}$ if the other endpoint is on 7. If the other endpoint is on r_6 we can turn the $2\frac{1}{2}$-jump into a $1\frac{1}{2}$-slope by removing an edge, and then the coverage of the remainder of r_6 ensures a crossing number of at least 2.

If there is a $\frac{1}{2}$-slope or 1-slope with an endpoint on s_2 next to a $2\frac{1}{2}$-jump, we can transform it into a jump, making the $2\frac{1}{2}$-jump a bar, and maintaining coverage. So any such configuration is isomorphic to a graph found in stage 1.

Any small structure that overlaps a rim branch covered by a large jump on the opposite side of the V_8 from that jump must also cover another whole rim branch. If it does not, then we can redefine the rim to go through that structure and remove the section of rim that is doubly covered. However if the overlapping structure does cover another rim branch as well, then we can remove the section of rim that is doubly covered and have a 2-crossing critical variant of $\mathcal{G}$, also yielding a 2-crossing graph.

It is also possible to have two large jumps on a single V_8. Some combinations of two large jumps yield a 2-crossing critical variant of $\mathcal{G}$, and therefore cannot occur in a critical graph. Other combinations fully cover the V_8 on their own, and still others leave graphs with 1-drawings. Since there are no three positioned large jumps for which any two yield a graph with crossing number 1, it is not possible to have more than two large jumps on a critical graph.

Therefore we can complete the collection of fully covered V_8's with large slopes by taking the pairs that give 1-drawings and completing the coverage with small structures.

At this stage, we should also consider the semi-diagonal. It was not useful in Stage 1, since by virtue of eliminating only one crossing pair the semi-diagonal cannot be in a fully covered V_8 with only small structures. If this were the case with a semi-diagonal from i to r_{i+4}, then the rim from $i-1$ to $i+1$ must be covered by those structures in order for the semi-diagonal to be useful, forcing the five sequential rim branches to be from $i+4$ to $i+1$. However given that situation, we can remove s_i and use the semi-diagonal as a spoke. If there is a structure with an endpoint v on s_i, then we remove $\langle v, s_i, i+4 \rangle$ instead, thereby preserving coverage. The resulting graph will still be fully covered, so the original graph was not 2-crossing critical.
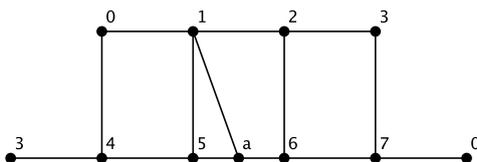

Figure 3.11: Semi-diagonal, or $3\frac{1}{2}$-jump

Since both the $2\frac{1}{2}$-jump and the off 3-jump leave a section of rim with just one possible crossing, a semi-diagonal could eliminate this crossing. However, with the semi-diagonal placed as shown in the figure we can use it as a spoke instead of s_1 and maintain coverage without the original spoke, thereby forcing any usage of the semi-diagonal to be non-critical.

After checking for isomorphism, there are 101 fully covered graphs with large jumps, 69 of which are not isomorphic to the graphs found in Stage 1.

3.3 Stage 3 – large slopes

This section introduces the last two structures into the discussion: the $1\frac{1}{2}$- and 2-slopes. A single 2-slope completely covers six of the eight rim branches, so to increase the crossing number to 2 we need only cover one of these two remaining rim branches. The $1\frac{1}{2}$-slope has a similar effect on the V_8, as shown in Figure 3.12. To get five sequential covered rim branches, we must cover

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either r_0 or both $\langle b, r_5, 6 \rangle$ and r_3. However, this coverage also places great limits on the criticality of the resulting 2-crossing graph.
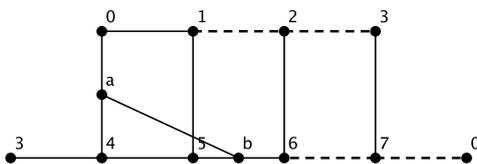

Figure 3.12: $1\frac{1}{2}$-slope

First we will examine ways to cover the remaining full rim branches with small structures. Any $\frac{1}{2}$-slope or 1-slope used can either be transformed into an off 1-jump or $1\frac{1}{2}$-jump without affecting the large slope, or we can remove the section of spoke under the small slope and still have a graph with crossing number 2. Consider three cases in covering a full rim branch: it is covered by a 1-jump, it is covered by two structures or it is fully covered by a structure that also covers another section of the rim. Covering with a 1-jump gives a 2-crossing critical graph. Suppose we cover one of the remaining rim branches with two of the small structures, say S_1 and S_2, each covering half of the rim branch. Then at least one of them, say S_1, also covers part of an adjacent rim branch R, otherwise the graph is either not critical or not 3-connected. If we redefine the rim to go through S_1 and delete R the rim branch is still covered in the resulting graph and so the original was not 2-crossing critical. The graph that results from this gives one of the other two cases.

The remaining case is when one of the rim branches is covered by a structure that also covers a disjoint section R of the rim: a $1\frac{1}{2}$-jump, 2-jump or bar. If the structure is a $1\frac{1}{2}$-jump or 2-jump, then deleting R leaves the 2-crossing critical graph $\mathcal{G}$. For a bar, deleting part of the spoke where the spoke and bar meet gives a 2-crossing critical graph. For a bar covering r_0, we delete $\langle a, s_0, 0 \rangle$ or, if the bar has an endpoint x on that part of the spoke, $\langle x, s_0, 0 \rangle$. If the bar covers r_3, we delete $\langle a, s_0, 4 \rangle$ or $\langle x, s_0, 4 \rangle$.

If the half rim branch is covered by a 2-jump, then this yields a 2-crossing critical graph without needing to cover anything else. This type of criticality will be considered in Section 4.2. If it is covered by a $\frac{1}{2}$-slope or 1-slope from s_2 to a point v on $[5, r_5, b\rangle$ then removing $\langle v, r_5, b \rangle$ leaves the Petersen graph. With a $\frac{1}{2}$-slope or 1-slope from s_1 to $\langle b, r_5, 6]$ we can remove $\langle 5, r_5, b \rangle$, leaving a $1\frac{1}{2}$-slope or 2-slope.

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Suppose there is a structure S with one endpoint, x, on $\langle 6, r_6, 7]$ and the other, y, on r_5 (S is either an off 1-jump or a $1\frac{1}{2}$-jump). Then we can redefine the rim to go through S with a spoke from 2 through 6 to y and delete $\langle 6, r_6, x\rangle$. If y is on $[b, r_5, 6\rangle$, then this leaves a $1\frac{1}{2}$-slope or 2-slope. Otherwise, y is on $\langle 5, r_5, b\rangle$ and the resulting graph is the Petersen graph, which is 2-crossing critical.

Alternatively, suppose there is a structure with one endpoint, x, on $[5, r_5, b\rangle$ and the other, y, on $\langle b, r_5, 6\rangle$. Again, we redefine the rim to go through S, this time deleting $\langle y, r_5, b\rangle$. This leaves a 2-slope or $1\frac{1}{2}$-slope.

This means that the only way to cover $\langle b, r_5, 6\rangle$ and maintain criticality is with a $\frac{1}{2}$-jump. The $1\frac{1}{2}$-slope with such a $\frac{1}{2}$-jump then gives the same coverage as a 2-slope.

We must also consider having two large slopes, or a large slope and large jump, on a V_8. Similarly to combining large jumps in Stage 2, we can have at most 2 large structures and those pairs that give 1-crossing graphs can have their coverage completed by small structures.

This gives a total of 19 non-isomorphic graphs, 12 of which were not found in Stages 1 and 2.

3.4   The Algorithm

In this section, we describe the algorithm we programmed to do the computations. All programs were written in the C programming language, using the graph structures of the `nauty` package as developed by Brenden McKay [7, 8]. The isomorphism test used is also from this package, and the planarity test from Boyer and Myrvold [4].

For Stage 1, we will cover rim branches r_0 through r_4 with small structures. There are 9 ways in which each of these rim branches can be covered: any of the 8 ways shown in Figure 3.9, or covered by a 2-bar. To find all possible combinations, we assign integers from 0 to 8 to each of the five rim branches, **rim**[0] through **rim**[4]. These numbers indicate the structures used to cover each rim branch, with 0 through 7 being the configurations shown in Figure 3.9 and 8 being a 2-bar.

Evidently, not all of these combinations will yield a 2-crossing critical graph, or indeed a graph. Any rim branch with an open jump to the right, that is

a value of 1, 3, 5 or 6, must be followed by a rim branch with an open jump to the left, a value of 2, 4, 5 or 6. Otherwise we are left with a partial edge having no second endpoint. A bar on r_0 requires a bar to also be on r_4, since they are in the same quad. Moreover, if any rim branch is covered by a 2-bar, then every other rim branch must also be covered by a 2-bar, as that structure covers all rim branches.

To ensure criticality, we must also take into account the restrictions discussed in Section 3.1. This limits the cases in which we may allow an open jump to the left on r_0 or to the right on r_4, thereby giving a structure that goes off the end of the 5 covered rim branches, as well an indicating where the second endpoint of such a jump can be placed. The placement of bars is also affected, both to ensure that no three sequential rim branches have bars, and to determine the placement of the endpoints of bars on adjacent quads.

Once these restrictions are observed, removing non-critical or impossible sets of values, we can construct the graphs that correspond to the remaining sets of values and check their criticality.

Stage 2 proceeds in a similar way after some initial setup. We first fix a large jump on the V_8. Small structures are used to cover the necessary remaining rim branches. For the $2\frac{1}{2}$-jump, there are three possible sets of additional rim coverage to give a 2-crossing critical graph, and for the off 3-jump there are two.

We use the same possible configurations to cover the required rim branches as were used in Stage 1. In any rim branch r_i with an endpoint of a large jump v in its interior, we treat the two sections $[i, r_i, v]$ and $[v, r_i, i+1]$ separately, with each receiving its own value and configuration of covering structures. Again, we must respect the restrictions found in Section 3.2 in choosing which of these sets of values are valid.

It is also important to consider a possible interaction of two large jumps on the same graph. To do this, we fix a jump on the V_8 and consider all possible placements of another jump on that V_8. A check can easily show if each pair of jumps gives 2-crossing critical graph that we can add to the list of critical graphs, a non-critical graph with crossing number 2 that we can discard, or a graph with a 1-drawing. For the few graphs of this final type, uncovered rim branches can take a configuration from Stage 1.

We follow the same process for Stage 3. The restrictions on placing small

structures with a large slope are such that only two possibilities exist for each slope, so those are easily constructed. We must then fix each slope and consider all possible placements of a large jump or another large slope on the same V_8. As we did with the two large jumps, we check for criticality and cover other rim branches with small structures as needed.

Chapter 4

Other Ways to Achieve Criticality

Fully covering a V_8 is not the only way to achieve a crossing number of 2. We have considered only the coverage caused by single structures, but the interaction of two such structures can sometimes create more coverage than the two structures individually, as the structures themselves can cross.

Oporowski found 201 2-crossing critical graphs with a V_8 minor but no V_{10}. Of these, all but the eight shown in Figure 4.1 are among the list of 312 found in Chapter 2. In examining these eight graphs, we can seek to understand the other ways in which criticality may be achieved.
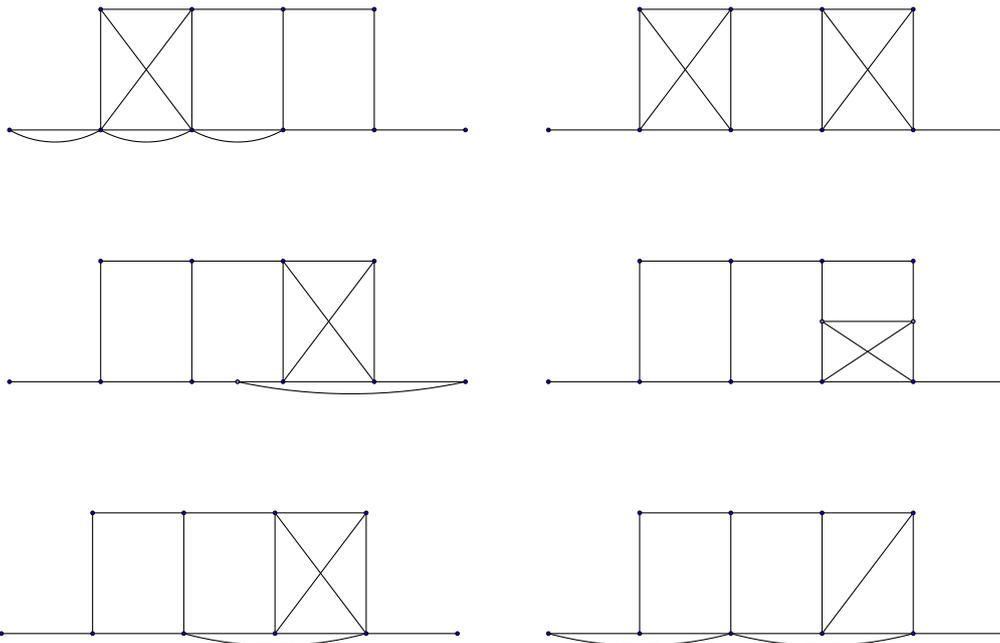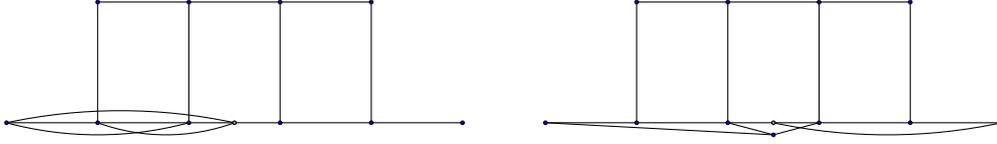
Figure 4.1: Oporowski's not fully covered graphs

In the first section, we examine the behaviour seen in the first four of these graphs where two structures cross inside a quad. The second section discusses graphs like the next three graphs, which have structures positioned so that the sections of rim they span overlap each other. In the final section, as in the final graph, we see an addition to the V_8 that is distinct from the structures we have previously discussed: a tree. In each of these sections, we provide a brief discussion of why the graphs are 2-crossing critical, and give a new example of a 2-crossing critical graph in that category.

4.1   Crossings in a Quad

In this section we consider pairings of structures that cross in a quad. Recall that a *quad* is the cycle formed by two consecutive spokes and the rim branches between them. Suppose we place two structures in a quad so that they cross, such as the two diagonals in Figure 4.2. Consider a drawing of the resulting graph in which no edge of the quad is crossed. Then the diagonals must be crossed in the drawing. There must also be a crossing on the V_8, since it is non-planar, so the drawing has at least 2 crossings.

Therefore any 1-drawing of a V_8 with structures crossed in a quad must include in its crossing an edge from the rim that forms part of the quad. By extension, this prevents all crossings involving the disjoint quad. We call the rim edges of the quad the *crossable edges*. For example, the crossable edges in Figure 4.2 are r_1 and r_5. Since any crossing must involve a crossable edge, covering r_1 also covers both r_4 and r_6.
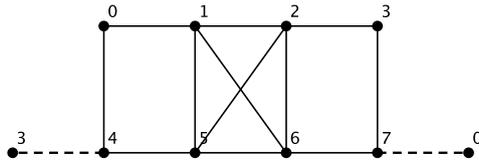
Figure 4.2: Crossed Diagonals

We must cover at least one of the crossable edges, otherwise we cannot have five sequential covered rim branches. There are two ways in which we can cover five in row: by covering three rim branches between the two covered by the crossed structures, such as those from 0 to 3, or by covering both of the crossable edges.

Let us explore in more detail the case of having two diagonals crossed in a quad and the ways to cover the remaining rim branches. If we use small structures to cover the required sections of rim and one of them overlaps the area already covered by the crossed diagonals, then we can remove the section of rim covered twice and still have a 2-crossing graph. In the case of an off 1-jump or $1\frac{1}{2}$ jump from a crossable edge to an adjacent edge, the remainder of the crossable edge must also be covered. We can remove the section of rim covered in the second edge and use the V_8 shown in Figure 4.3 to yield a fully covered graph. A 2-jump that covers one of the crossable edges gives a 2-crossing graph without any other coverage, as will be discussed in Section 4.2. This means that, among the small structures, the only one useful for fully covering with crossed diagonals is a 1-jump.
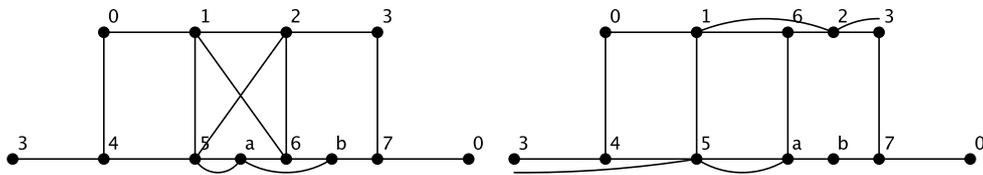

Figure 4.3: Crossed Diagonals with an off 1-jump

Using large structures, we can achieve criticality by placing a $2\frac{1}{2}$-jump, off 3-jump or 2-slope so that it covers both crossable edges thereby completing coverage. Placing a second pair of crossed diagonals in the quad non-adjacent to the first pair also gives critical full coverage, with each pair preventing all crossing in the quad containing the other.

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Diagonals are not the only structures able to give this kind of crossing; we can also consider semi-diagonals, bars, $\frac{1}{2}$-slopes and 1-slopes. Any two of these crossed in a quad force any crossing in a 1-drawing to involve at least one of the rim branches of that quad. We note, however, that some of these structures also cover at least part of those rim branches. For example, any crossing in a quad involving a bar must yield a 2-crossing graph since the bar itself covers both crossable edges.
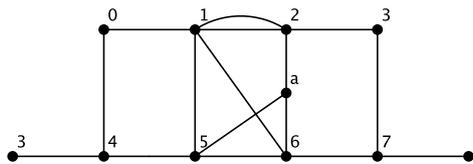

Figure 4.4: A new 2-crossing critical graph with a crossing in a quad

In fact, we can consider the $1\frac{1}{2}$-slope and 2-slope as following the same pattern if we consider it in conjunction with the spoke it crosses. Suppose the slope has an endpoint on s_i and crosses s_{i+1}. If we have a drawing of the graph in which the quad defined by those spokes is not crossed, then the slope must be crossed with some edge. Since there must also be a crossing on the V_8, this gives a drawing with at least 2 crossings. This, together with the slope covering the rim it spans, account for the coverage provided by the large slopes.

4.2   Accessibility

In this section, we introduce and begin the analysis of the concept of accessibility. The *inside* of a 1-drawing of a V_8 is the area enclosed by the rim and spokes, as shown by the shaded region in Figure 4.5. The remaining faces are the *outside*. We say that a vertex u is *accessible* from a vertex v if, in a 1-drawing of the V_8, we can place an edge $\{u, v\}$ that crosses no other edge and is entirely inside the V_8.
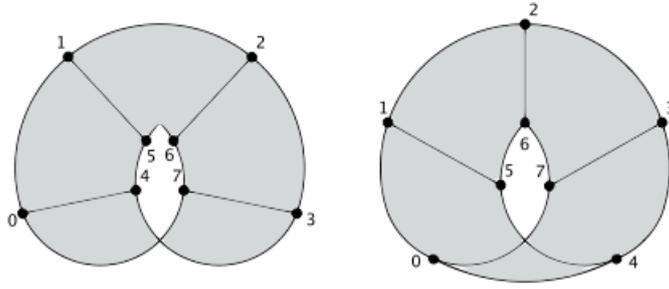
Figure 4.5: The inside and outside of a V_8 drawing

When working with the V_8 alone, and no restrictions on which edges can be crossed, a vertex on the graph can access most others. The dotted lines and open points in Figures 4.6 and 4.7 indicate the positions accessible from the vertex x, represented by the diamond in the figures. This can easily be seen by considering the finite number of possible drawings and the area accessible in each. As we add structures, limiting crossings and creating possible obstacles for new edges, the accessibility will change. It is worth noting here that any of the structures we have defined can be drawn inside the V_8 with the exception of the off 2-jump, 2-jump and 2-slope. These structures must be outside the V_8 in any 1-drawing, although we can choose a new V_8 to turn the off 2-jump into a bar instead.
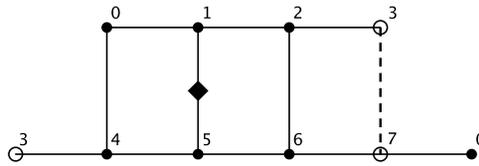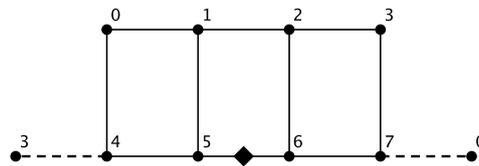

Figure 4.6: Inside access from a spoke

Figure 4.7: Inside access from a rim branch

Let us consider how a structure can change accessibility and possibly force a second crossing in our graph. Suppose we have a 2-jump from i to $i + 2$.

The 2-jump also covers the rim branches it spans, preventing all crossings involving them. Moreover, we know that the 2-jump must be outside the V_8, so any structure with an endpoint on r_i, $i + 1$, r_{i+1} or s_{i+1} must be inside. The access to a vertex on these edges is limited by the 2-jump, as shown in Figures 4.8 and 4.9.
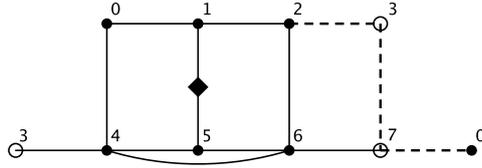

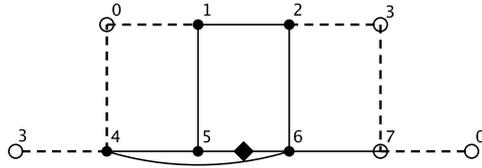Figure 4.8: Inside access from a spoke

Figure 4.9: Inside access from a rim branch

We must note that a lack of access is not necessarily enough to force a second crossing, since we must consider other possible V_8's. For example, if there is an off 1-jump from a vertex u on r_{i+1} to v on r_{i+2}, then consider the V_8 with a spoke from $i+6$ through $i+2$ to u, and a rim branch from v through u to $i + 3$. This gives a V_8 with a 2-slope and $\frac{1}{2}$-slope from one of its spokes, which we know to have a 1-drawing. It is only with the V_8 as originally defined that the two structures are forced to cross.
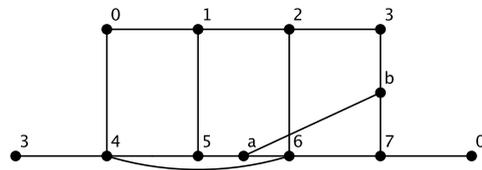

Figure 4.10: A new 2-crossing critical graph with overlapping structures

Even if the original structure does not necessarily have to be on the outside of the V_8, it can cause a similar situation. A $1\frac{1}{2}$-jump from i to r_{i+1} can be drawn inside a V_8, but only the section of r_{i+1} not spanned by the jump crosses

r_{i+4}. However, if the remainder of r_{i+1} is covered by some structure, then access from vertices on r_i, r_{i+1} and s_{i+1} is limited in much the same way as it would be if a 2-jump was placed there instead. In this way, the overlap of structures can force a second crossing and possibly give a 2-crossing critical graph.

4.3   Trees

In this section, we consider adding a tree to a V_8. Up to this point, the only structures we have added to the V_8 are single edges with both endpoints in the V_8. It is also possible to have vertices disjoint from the V_8. One way in which this can occur is by taking a small tree and identifying its leaves with vertices on the V_8. The simplest form of this is attaching a star, an example of which can be seen in Figure 4.11.
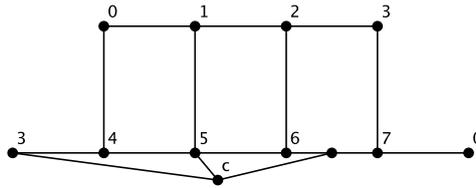

Figure 4.11: A tree attached to a V_8

Take any pair of leaves, u and v, of the tree being attached. Then the tree must cover at least those sections of rim covered by a structure with endpoints u and v. Therefore the tree in Figure 4.11 must cover the rim at least from 3 to u, since it forms a 2-jump from 3 to 5 and a $1\frac{1}{2}$-jump from 5 to u. More than this, however, it also covers from 0 to 2. These two rim branches are covered for different reasons: r_0 because all the rim branches it could have crossed are covered, and r_1 because crossing the remaining section it could have crossed, from u to 7, forces a $1\frac{1}{2}$-jump from 5 to u to be drawn inside, which cannot happen while it is attached at v.
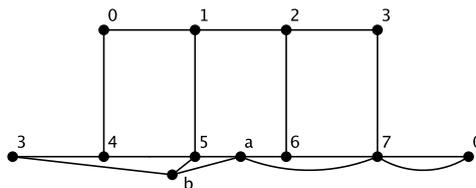

Figure 4.12: A new 2-crossing critical graph with a tree

There are, of course, limits on where attaching such trees can be useful. Any tree with leaves on rim branches r_i and r_{i+4} creates a V_{10} minor, which puts the graph outside our realm of exploration. A tree that attaches to spokes s_i and s_{i+2} creates a two bar with the path between those leaves, so any other branches cause the graph to be non-critical.

As the number of leaves on the tree increases, so do the number of possible layouts for the tree. A tree with 4 leaves may be in the form of a star, or it may divide the leaves into two pairs attached to vertices v and v', with an additional edge between these two new vertices.

Chapter 5

Conclusion

The problem of classifying all 2-crossing critical graphs remains unsolved, but the class of unknown graphs is getting smaller. Using our definition of a fully covered V_8, we have found 312 non-isomorphic 2-crossing critical graphs. It remains to find the 2-crossing critical graphs with a V_8 minor, but no V_{10} minor, that are not fully covered.

The structures of the remaining known graphs are a part of the classes discussed in Chapter 4. Clearly Oporowski's are not the only graphs to fit into these classes, as we have seen other examples. These classes need to be more thoroughly examined in order to provide an exhaustive list.

Moreover, it remains to be shown whether or not these are the only classes of 2-crossing critical graphs with a V_8 minor but no V_{10}. All graphs we encountered fit into one of the classes, but that is far from a conclusive proof.

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