Combinatorics and the KP Hierarchy

by

Sean Carrell

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The study of the infinite (countable) family of partial differential equations known as the Kadomtzev - Petviashvili (KP) hierarchy has received much interest in the mathematical and theoretical physics community for over forty years. Recently there has been a renewed interest in its application to enumerative combinatorics inspired by Witten's conjecture (now Kontsevich's theorem).

In this thesis we provide a detailed development of the KP hierarchy and some of its applications with an emphasis on the combinatorics involved. Up until now, most of the material pertaining to the KP hierarchy has been fragmented throughout the physics literature and any complete accounts have been for purposes much different than ours.

We begin by describing a family of related Lie algebras along with a module on which they act. We then construct a realization of this module in terms of polynomials and determine the corresponding Lie algebra actions. By doing this we are able to describe one of the Lie group orbits as a family of polynomials and the equations that define them as a family of partial differential equations. This then becomes the KP hierarchy and its solutions. We then interpret the KP hierarchy as a pair of operators on the ring of symmetric functions and describe their action combinatorially. We then conclude the thesis with some combinatorial applications.

Acknowledgments

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$\check{f}$	Contraction operator	25
χ^λ_ρ	Irreducible character indexed by λ evaluated at a permutation with cycle type ρ	13
$\ell(\lambda)$	Length of λ	6
$\hat{F}^{(m)}$	Formal completion of $F^{(m)}$	39
$\hat{r}^B_m, \hat{R}^B_m$	Transported representation via σ_m	35
$\hat{r}_m$	Representation of $\overline{a_\infty}$ and a_∞	24
$\hat{v}$	Wedging operator	25
λ'	Conjugate of λ	7
λ/μ	Skew diagram λ/μ	7
$\lambda^{(-i)}$	Partition obtained from λ by switching the ith U (from the right) to a R	49
$\lambda^{(i)}$	Partition obtained from λ by switching the ith R (from the left) to a U	47
λ_k	kth Chern class of the Hodge bundle	67
ω	Fundamental involution	10
$\overline{a_\infty}$	Infinite band matrix	21
ψ_i	First Chern class of $\mathcal{L}_i$	67
ψ_m	Charge m vacuum vector	18
σ	Boson - Fermion isomorphism	35
$\widehat{H}$	Analog of H on F	66
$\widehat{J}$	Join-cut operator on F	66
$\widetilde{G} = \widetilde{G}(u;,T_0,T_1,\cdots)$	Enriched Hodge integral generating series	67
$\widetilde{H} = \widetilde{H}(z,p)$	Disconnected simple Hurwitz number generating function	64
a_∞	Central extension of $\overline{a_\infty}$	25
e_λ	Elementary symmetric functions	9
$g\ell_\infty$	Infinite matrix	20
h_λ	Complete symmetric functions	9
$h_k(\alpha)$	Simple Hurwitz number	64
$m_i = m_i(\lambda)$	Number of times i occurs as a part of λ	6
p_λ	Power sum symmetric functions	11
$r(a)$	Representation of $g\ell_\infty$	21
r_m, R_m	Restriction of r and R to $F^{(m)}$	22
s_λ	Schur function	12
$u_i(\lambda)$	Number of up-steps that follow the ith right-step	47
v_λ	Semi-infinite monomial associated to λ	20
$\mathbf{cyc}(\sigma)$	Cycle type of σ	13

Chapter 1

Introduction and Background

The study of the infinite (countable) family of partial differential equations known as the Kadomtzev - Petviashvili (KP) hierarchy has received much interest in the mathematical and theoretical physics community for over forty years. Recently there has been a renewed interest in its application to enumerative combinatorics inspired by Witten's conjecture [32] (now Kontsevich's theorem [19]).

The purpose of this thesis is to provide a detailed development of the KP hierarchy and some of its applications with an emphasis on the combinatorics involved. Up until now, most of the material pertaining to the KP hierarchy has been fragmented throughout the physics literature and any complete accounts have been for purposes much different than ours.

In the remainder of this section we give a very brief account of the KP hierarchy, culminating in an overview of Witten's conjecture and some of the more recent applications of the KP hierarchy to enumerative algebraic combinatorics. For additional material on the KP hierarchy and related items see [24], [23].

1.1 A Historical Introduction

The history of the KP hierarchy begins with the search for a certain class of solutions to a (1+1)-dimensional nonlinear wave equation called the Korteweg-de Vries (KdV) equation (see [23], [12] or [13]). Here we use (1+1)-dimensional to denote a partial differential equation in two variables, one of which represents time and the other a spatial variable.

Explicitly, the *KdV equation* is the following partial differential equation for some complex function $u = u(x, t)$ where we use subscripts to denote partial differentiation with respect to the given variables:

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.1}$$

A particularly useful approach to solving nonlinear partial differential equations, which is connected to the main topic of this thesis, is the Lax method. The idea is to construct a linear problem in which the compatibility relations between a pair of linear differential operators gives rise to the original nonlinear problem.

Let L be a linear ordinary differential operator in the spatial variable x. We allow L to have coefficients that are functions in the spatial variable x and a time variable t. We are interested in the spectral problem for L. That is, we wish to find an eigenfunction $w = w(x,t)$ and the associated eigenvalue $\lambda = \lambda(t)$ such that

$$Lw = \lambda w. \tag{1.2}$$

In general, λ will depend on the time variable t. We will further restrict the problem to the case where λ has no time dependence (i.e. it is constant). We may write this as

$$\lambda_t = 0. \tag{1.3}$$

The problem described in (1.2) and (1.3) is often called the isospectral problem.

In what follows we will use the notation L_t to denote the linear ordinary differential operator obtained from L by differentiating each coefficient in L by t.

Equation (1.2) tells us something about how w behaves in the spatial coordinate, but since $\lambda_t = 0$, it tells us nothing about the time evolution of w. We will assume that w evolves in time with respect to a linear ordinary differential operator in the spatial variable x called B. As in the case of L, we allow B to have coefficients that are functions in both x and t. In other words, we assume that

$$w_t = Bw. \tag{1.4}$$

Differentiating (1.2) with respect to t gives

$$L_t w + L w_t = \lambda_t w + \lambda w_t.$$

Using (1.3) and (1.4), this becomes

$$\begin{aligned} L_t w + LBw &= \lambda Bw \\ &= B\lambda w \\ &= BLw, \end{aligned}$$

or in other words,

$$L_t w = [B, L]w, \tag{1.5}$$

where here $[B, L] = BL - LB$ is the standard commutator bracket.

It can also be shown that (1.5) implies

$$L_t = [B, L], \tag{1.6}$$

and so we have an equality of linear ordinary differential operators in x. Taking coefficients on each side gives a set of equations for the coefficients of the L and B operators. If the original nonlinear partial differential equation appears as one of the equations for the coefficients of the L and B then we say that (1.6) is the Lax form of the nonlinear partial differential equation.

For example, let

$$L = \frac{\partial^2}{\partial x^2} + u \tag{1.7}$$

be the Schrödinger operator with potential u where u is a function of x and t. By a change of variable, we can always transform a linear differential operator of order n to one that is monic and whose order $(n-1)$ component is zero.

The first case in which (1.6) is non-trivial for L given by (1.7) occurs when B is a third order operator. In this case, for functions b_1 and b_0,

$$B = \frac{\partial^3}{\partial x^3} + b_1 \frac{\partial}{\partial x} + b_0.$$

Then (1.6) gives us the following set of equations that must be satisfied:

$$\begin{aligned} b_1 &= \frac{3}{2} u, \\ b_0 &= \frac{3}{4} u_x, \\ u_t &= \frac{3}{2} u u_x + \frac{1}{4} u_{xxx}. \end{aligned}$$

After scaling x and t, the condition on u is that it satisfies the KdV equation (1.1).

In fact, for each positive $2n+1$ there is an operator B_{2n+1} that satisfies (1.6) and it turns out that each of these operators gives rise to an integral of motion for the KdV equation. Under (1.6) these become the so-called "higher KdV equations" [21].

It is also known that the KdV equation (among some other nonlinear partial differential equations such as the KP equation discussed below) admits a family of special solutions called *soliton* solutions. These are solutions that for $t \to \pm\infty$ look like superpositions of solitary wave solutions. As t moves from $+\infty$ to $-\infty$ (or $-\infty$ to $+\infty$) the solitary waves come together and interact with the special property that after the interaction occurs, the original solitary waves are preserved up to a phase change. Since these solutions have properties that are unexpected such as having a type of superposition principle there is much interest in finding and constructing them.

In [11, 12], Hirota introduced a method to find these soliton solutions. Hirota's method involves constructing a bilinear form of the partial differential equation in question in order to test various soliton equation ansätze. In particular, Hirota discovered that if you make the change of variables

$$u(t,x) = -2 \frac{\partial^2}{\partial x^2} \ln f(t,x)$$

then the KdV equation becomes

$$ff_{xt} - f_t f_x + f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^2 = 0,$$

a homogeneous quadratic equation. A differential equation in this form is sometimes called a bilinear differential equation. Hirota introduced a method that allowed one to find exact solutions to bilinear differential equations using a perturbation method. This allowed Hirota to construct N-soliton solutions for any N (these solutions look like the superposition of N solitons).

After Hirota's bilinear method was introduced there were a large number of results published by a group of Japanese physicists at Kyoto University consisting of M. Sato, T. Miwa, M. Jimbo, M. Kashiwara, E. Date and Y. Sato. This group is sometimes referred to as the Sato school or the Kyoto school.

The Kyoto school found that not only are all the higher KdV equations part of a single hierarchy of mutually commuting flows on a space of functions (the *KdV hierarchy*) but that in fact they formed a sub-hierarchy of a family of equations known as the *KP hierarchy*, so called because the simplest equation is the *KP equation*

$$\frac{3}{4}\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} - \frac{3}{2}u\frac{\partial u}{\partial x} - \frac{1}{4}\frac{\partial^3 u}{\partial x^3}\right). \tag{1.8}$$

The method of constructing this hierarchy of related partial differential equations was found through the use of the Lax form applied to pseudo-differential operators rather than just ordinary differential operators [5]. Also, it was discovered that the Lax form was connected to Hirota's bilinear form through the use of a special function called the tau (or τ) function [23]. Tau functions were originally used in the context of holonomic quantum field theory [29]. Afterwards it was discovered that in the context of integrable systems, the tau functions completely encoded solutions of the KP hierarchy. In addition, there exists a bilinear form that each of these tau functions satisfy and that completely characterizes the KP hierarchy [16].

In studying solutions to the KP hierarchy, M. Sato found that Schur functions appeared as tau functions. This led him to consider a linear group action on the space of tau functions, and thus led to the description of the space of tau functions for the KP hierarchy as a connected cell in a particular infinite dimensional Hilbert space Grassmannian [28]. In fact, each point in this cell can be characterized by a single function which can be easily transformed into a tau function for the KP hierarchy. For a complete mathematical description see Segal and Wilson [30, 27].

While considering representation theory using vertex operators, Miwa, Jimbo, Date and Kashiwara noticed that the bilinear form of the KP hierarchy consisted of the vertex operator representation of the infinite Lie group GL_∞ [4]. This prompted them to construct the solutions to the KP hierarchy from a representation theory point of view. This method is covered in their series of papers beginning with [16]. See also [23], [15] and [14]. In this thesis we will make use of the representation theoretic description of the KP hierarchy through vertex operators.

We now turn our attention to some combinatorial geometric applications of the KP hierarchy. One such application which has renewed interest in the KP hierarchy from a combinatorial geometric perspective was Witten's conjecture concerning the generating function for certain intersection numbers on the moduli space of marked curves. Witten's conjecture has since been proven in a few ways, the first of which was by Kontsevich [19].

Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne-Mumford compactification of the moduli space of genus g curves with n marked points. If we let $X = (\mathcal{C}, x_1, x_2, \cdots, x_n)$ be a point in $\overline{\mathcal{M}}_{g,n}$ where $\mathcal{C}$ is a genus g curve and $x_1, \cdots, x_n$ are the marked points then we can associate with each marked point the line bundle $\mathcal{L}_i$ on $\overline{\mathcal{M}}_{g,n}$ whose fiber at the point X is the cotangent line to $\mathcal{C}$ at x_i. Then we let ψ_i be the first Chern class of $\mathcal{L}_i$ and we denote the intersection number by

$$\langle \tau_{m_1} \cdots \tau_{m_n} \rangle = \int_{\overline{M}_{g,n}} \psi_1^{m_1} \cdots \psi_n^{m_n}.$$

For complete details see Lando and Zvonkin [20].

If we now construct the generating function of intersection numbers

$$F(t_0, t_1, \cdots) = \left\langle \exp\left(\sum_i t_i \tau_i \right) \right\rangle$$

then Witten's conjecture says that $\exp(F)$ is a tau function for the KdV hierarchy.

An alternate form of Witten's conjecture says that

$$V = \frac{\partial^2}{\partial t_0^2} F$$

is the partition function for the universal one matrix model. This has combinatorial interest since it is known (see [2]) that matrix models can be thought of as certain weighted sums of maps (graphs embedded on surfaces). This is in fact the way in which Kontsevich originally proved Witten's conjecture, by showing that V could be described as the evaluation of a certain operator over all metric fat-maps (see [19]).

After Kontsevich proved Witten's conjecture, T. Ekedahl, S. K. Lando, M. Shapiro and A. Vainshtein published a formula known as the ELSV formula, which related intersection numbers as described above and Hurwitz numbers. Briefly, Hurwitz numbers enumerate ramified coverings of the sphere with arbitrary ramification at a single point and simple ramification elsewhere. Combinatorially however, the Hurwitz number $H(k, \sigma)$, for $\sigma \in \mathcal{S}_n$ a permutation of n and k a positive integer, is the number of ways of factoring σ into a product of k transpositions where the k transpositions generate a transitive subgroup of $\mathcal{S}_n$.

Shortly afterwards, Okounkov showed that the generating function for Hurwitz numbers satisfied the KP hierarchy and, using some additional theory involving matrix integrals and asymptotics, showed that this implied Witten's conjecture [26]. More recently, Kazarian and Lando [18] have given a more direct proof of Witten's conjecture using only ELSV and the fact that the Hurwitz numbers satisfy the KP hierarchy.

Since being a tau function for the KP hierarchy implies that the function satisfies a family of partial differential equations, there is much interest in finding other combinatorial generating functions which are also tau functions. Work in this direction by Goulden and Jackson [9] has resulted in many interesting results including an application (see Bender, Gao and Richmond [1]) to the asymptotics of triangulations.

1.2 Partitions

In this section and the next we briefly review some facts about partitions and the ring of symmetric functions. For the most part we follow Macdonald [22] and full details and proofs can be found there or in Stanley [31].

A *partition* is any (finite or infinite) sequence

$$\lambda = (\lambda_1, \lambda_2, \cdots)$$

of non-negative integers in non-increasing order

$$\lambda_1 \geq \lambda_2 \geq \cdots$$

where at most finitely many λ_i are non-zero. The λ_i are called the *parts* of the partition λ. We use ϵ to denote the empty partition that has all parts equal to zero.

Throughout this thesis we will use alternate interpretations of the above definition, sometimes viewing partitions as consisting of only the non-zero parts and other times viewing them as countable sequences with finitely many non-zero entries. Both descriptions are equivalent and which version we are using will be clear from the context.

Given a partition λ, we say that the *length* of the partition, $\ell(\lambda)$, is equal to the number of non-zero parts and the *size* of the partition, $|\lambda|$, is the sum of the parts

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots .$$

If λ is a partition of size n then we write $\lambda \vdash n$. Sometimes it is convenient to use the following notation for a partition λ:

$$\lambda = (1^{m_1} 2^{m_2} \cdots),$$

where $m_i = m_i(\lambda)$ is the number of times that i occurs as a part of λ and is called the *multiplicity* of i in λ. We may also write a partition in this form with the size of the parts in decreasing rather than increasing order, as in example 1.2.1 below.

We sometimes represent a partition as a connected collection of unit squares on the integer lattice. For a partition $\lambda = (\lambda_1, \lambda_2, \cdots)$ we say that its (Young) *diagram* is a collection of unit squares in the integer lattice, left aligned such that the first row consists of λ_1 squares, the second row consists of λ_2 squares, etc.

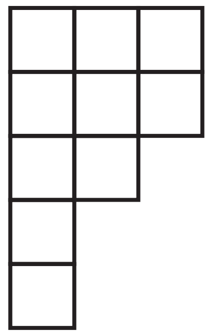

Figure 1.1: Young Diagram Example

Example 1.2.1. *Consider the partition $\lambda = (3, 3, 2, 1, 1) = 1^2 2 3^2 = 3^2 2 1^2$. This partition has the Young diagram in Figure 1.1 and is such that $|\lambda| = 10$, which is the number of squares in Figure 1.1, and $\ell(\lambda) = 5$, which is the number of rows in Figure 1.1.*

We will often use λ to denote both the partition and its diagram with the usage clear from the context.

For a partition λ we form its *conjugate*, denoted by λ', by taking the diagram of λ and reflecting in the main diagonal. The partition corresponding to this new diagram is the conjugate. From this description it is easy to see that the parts of λ' are the lengths of the columns of λ, or in other words

$$\lambda'_i = |\{j : \lambda_j \geq i\}|.$$

For any two partitions λ, μ we say that $\mu \subseteq \lambda$ if the diagram of μ fits inside the diagram of λ, or in other words $\lambda_i \geq \mu_i$ for all i.

Let λ, μ be partitions such that $\mu \subseteq \lambda$. If we draw the diagram for λ and then remove the squares corresponding to the diagram of μ, what we are left with is called the *skew diagram* λ/μ. If $\theta = \lambda/\mu$ is a skew diagram then $\theta_i = \lambda_i - \mu_i$ is the number of squares in the ith row of the skew diagram. We let $\theta' = \lambda'/\mu'$ be the conjugate of the skew diagram θ and as such $\theta'_i = \lambda'_i - \mu'_i$ is the number of squares in the ith row of the conjugate skew diagram θ' and hence the number of squares in the ith column of the skew diagram θ.

A skew diagram $\theta = \lambda/\mu$ is called a *horizontal m-strip* if $|\theta| = m$ and $\theta'_i \leq 1$ for each i. In other words, θ is a horizontal m-strip if it contains m squares and no two squares are vertically adjacent. Similarly we say that θ is a *vertical m-strip* if $|\theta| = m$ and $\theta_i \leq 1$ for each i.

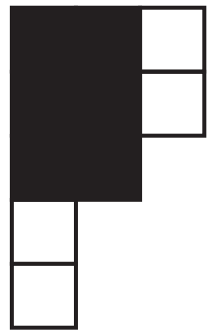

Figure 1.2: Skew Diagram Example

Example 1.2.2. *Let $\lambda = 3^2 21^2$ and let $\mu = 2^3$. Then $\mu \subseteq \lambda$ and we can form the skew diagram $\theta = \lambda/\mu$. This can be seen in Figure 1.2 where the partition formed by the black and white squares is λ, the partition formed by just the black squares is μ and the skew diagram θ is the set of squares that are white. Also, notice that λ/μ is a vertical 4-strip.*

1.3 Symmetric Functions

Let $\mathcal{S}_n$ be the group of permutations on n elements. The group $\mathcal{S}_n$ is also called the *symmetric group* on n elements. Thus, $\mathcal{S}_n$ is the set of bijections from $\{1, \cdots, n\}$ to itself with composition as the group operation. Let $\mathbb{Q}[x_1, \cdots, x_n]$ be the ring of polynomials in the n algebraically independent variables $x_1, \cdots, x_n$ with rational coefficients. There is a natural action of $\mathcal{S}_n$ on $\mathbb{Q}[x_1, \cdots, x_n]$ where $\sigma \in \mathcal{S}_n$ takes $p(x_1, \cdots, x_n) \in \mathbb{Q}[x_1, \cdots, x_n]$ to $\sigma p = p(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$. We say that a polynomial $p \in \mathbb{Q}[x_1, \cdots, x_n]$ is a *symmetric polynomial* if $\sigma p = p$ for all $\sigma \in \mathcal{S}_n$. We let Λ_n be the ring of all symmetric polynomials in n variables.

The ring Λ_n is a graded ring,

$$\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k,$$

where Λ_n^k contains polynomials $p \in \Lambda_n$ which are homogeneous of total degree k, along with the zero polynomial.

For integers m, n such that $m \geq n$, the homomorphism

$$\mathbb{Q}[x_1, \cdots, x_m] \to \mathbb{Q}[x_1, \cdots, x_n]$$

formed by taking $x_{n+1}, \cdots, x_m$ to zero and all other x_i to themselves restricts to a homomorphism

$$\Lambda_m^k \to \Lambda_n^k$$

and so we may form the inverse limit

$$\Lambda^k = \varprojlim_n \Lambda_n^k.$$

We call the ring

$$\Lambda = \bigoplus_{k \geq 0} \Lambda^k$$

the ring of *symmetric functions*.

We will now describe four families of symmetric functions, each of which is a basis for Λ and will be important later on.

Every symmetric function is a formal infinite sum of monomials in a countable family of indeterminates. For ease of notation we will suppress these variables when possible. Also, we will occasionally use an unadorned variable name to refer to the vector consisting of all the variables. For example, we will write x to mean $x = (x_1, x_2, \cdots)$ so that we can write $f(x) \in \Lambda$ when we wish to make clear which family of indeterminates we are considering.

For each $r \geq 0$ the rth *elementary symmetric function* e_r is defined by the generating function

$$E(t) = \sum_{r \geq 0} e_r t^r = \prod_{i \geq 1} (1 + x_i t) \tag{1.9}$$

where t is distinct from the x_i.

Example 1.3.1. *The first few elementary symmetric functions are*

$$\begin{aligned} e_0 &= 1, \\ e_1 &= \sum_i x_i, \\ e_2 &= \sum_{i<j} x_i x_j, \\ e_3 &= \sum_{i<j<k} x_i x_j x_k. \end{aligned}$$

In general, the elementary symmetric functions have the form

$$e_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}. \tag{1.10}$$

Furthermore, for each partition λ we define

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots .$$

It is known that the e_r are algebraically independent and that they generate Λ.

For each $r \geq 0$ the rth *complete symmetric function* h_r is defined by the generating function

$$H(t) = \sum_{r \geq 0} h_r t^r = \prod_{i \geq 1} \frac{1}{1 - x_i t}, \tag{1.11}$$

and similarly to the elementary symmetric functions, we define

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots .$$

Example 1.3.2. *The first few complete symmetric functions are*

$$\begin{aligned} h_0 &= 1, \\ h_1 &= \sum_i x_i = e_1, \\ h_2 &= \sum_{i \le j} x_i x_j, \\ h_3 &= \sum_{i \le j \le k} x_i x_j x_k. \end{aligned}$$

In general, the complete symmetric functions have the form

$$h_n = \sum_{i_1 \le \cdots \le i_n} x_{i_1} \cdots x_{i_n}. \tag{1.12}$$

From the generating functions (1.9) and (1.11) we have

$$H(t)E(-t) = 1$$

or

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0 \qquad \forall n \ge 1. \tag{1.13}$$

Since the e_r are algebraically independent we may define a homomorphism of graded rings

$$\omega : \Lambda \to \Lambda$$

by

$$\omega(e_r) = h_r.$$

From (1.13) we see that ω is an involution and hence an automorphism of Λ and so the h_r are also an algebraically independent set of generators for Λ. This involution is often called the *fundamental involution*.

For each $r \ge 1$ the rth *power sum symmetric* function is

$$p_r = \sum_i x_i^r$$

and $p_0 = 1$. We can construct the corresponding generating function

$$\begin{aligned} P(t) &= \sum_{r \ge 1} p_r t^{r-1} \\ &= \sum_{i \ge 1} \sum_{r \ge 1} x_i^r t^{r-1} \\ &= \sum_{i \ge 1} \frac{x_i}{1 - x_i t} \\ &= \sum_{i \ge 1} \frac{\partial}{\partial t} \log \frac{1}{1 - x_i t}. \end{aligned}$$

From this we get

$$\begin{aligned} P(t) &= \frac{\partial}{\partial t} \log \prod_{i \geq 1} \frac{1}{1 - x_i t} \\ &= \frac{\partial}{\partial t} \log H(t) \\ &= \frac{H'(t)}{H(t)} \end{aligned} \tag{1.14}$$

and likewise

$$P(-t) = \frac{\partial}{\partial t} \log E(t) = \frac{E'(t)}{E(t)}. \tag{1.15}$$

From (1.14) and (1.15) we get

$$\begin{aligned} nh_n &= \sum_{r=1}^{n} p_r h_{n-r} \\ ne_n &= \sum_{r=1}^{n} (-1)^{r-1} p_r e_{n-r} \end{aligned}$$

and so we see that the p_r are algebraically independent and generate Λ. Furthermore, for any partition λ we define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$$

and all such p_λ form a linear basis for Λ.

Since ω changes e_r to h_r it follows from (1.14) and (1.15) that

$$\omega(p_n) = (-1)^{n-1} p_n$$

and hence for any partition λ

$$\omega(p_\lambda) = \epsilon_\lambda p_\lambda$$

where

$$\epsilon_\lambda = (-1)^{|\lambda| - \ell(\lambda)}.$$

In order to describe the elementary and complete symmetric functions in terms of the power sum symmetric functions we let

$$z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!$$

where $m_i = m_i(\lambda)$ is the number of parts of λ equal to i. Then

$$\begin{aligned} H(t) &= \exp\left(\sum_{r \geq 1} \frac{p_r t^r}{r} \right) \\ E(t) &= \exp\left(- \sum_{r \geq 1} \frac{(-1)^r p_r t^r}{r} \right) = \frac{1}{H(-t)}, \end{aligned}$$

or equivalently,

$$h_n = \sum_{|\lambda|=n} z_\lambda^{-1} p_\lambda \tag{1.16}$$
$$e_n = \sum_{|\lambda|=n} \epsilon_\lambda z_\lambda^{-1} p_\lambda.$$

The last family of symmetric functions that we will make use of is the Schur functions. For any partition λ the *Schur function* s_λ is defined as

$$\begin{aligned} s_\lambda &= \det\left(h_{\lambda_i - i + j}\right)_{1 \le i,j \le \ell(\lambda)} \\ &= \det\left(e_{\lambda'_i - i + j}\right)_{1 \le i,j \le \ell(\lambda')} \end{aligned} \tag{1.17}$$

where the second equality is a relatively routine calculation which can be found in Macdonald [22]. Here we use the convention that $h_i = 0$ (or $e_i = 0$) if $i < 0$.

From (1.17) we see that

$$\omega(s_\lambda) = s_{\lambda'}$$

and also that

$$s_{(n)} = h_n, \qquad s_{(1^n)} = e_n.$$

It is also the case that the Schur functions form a linear basis for Λ. Note that since the fundamental involution acts on Schur functions by taking s_λ to $s_{\lambda'}$ we sometimes use ω to mean the involution on the set of partitions of n that takes λ to λ'.

At this point we would like to introduce a different way of describing Schur functions which is common in the physics literature. Since $\Lambda = \mathbb{Q}[p_1, p_2, \cdots]$ and each of the p_r are algebraically independent we may construct the isomorphism

$$\begin{aligned} \Phi &: \Lambda \to \mathbb{Q}[y_1, y_2, \cdots] \\ &: p_n \mapsto n y_n \end{aligned}$$

where the y_i form a distinct set of indeterminates.

Under Φ the resulting polynomials

$$S_\lambda = \Phi(s_\lambda)$$

are called the *Schur polynomials* and in particular, for n a positive integer, the

$$S_n = S_{(n)} = \Phi(s_{(n)}) = \Phi(h_n)$$

are called the elementary Schur polynomials and have the generating function

$$\sum_n S_n t^n = \exp\left(\sum_r y_r t^r\right) \tag{1.18}$$

which is simply the image of the generating function for the complete symmetric functions under the isomorphism Φ. Since the s_λ form a basis for Λ, the S_λ form a basis for $\mathbb{Q}[y]$ and so every polynomial can be written in terms of the Schur polynomials (here we really do mean polynomials and not symmetric polynomials). It will be useful for us to switch between these two points of view but it should be clear from the context and language which point of view we are using.

Heading back to symmetric function theory, we define a bilinear form $\langle \cdot, \cdot \rangle$ on Λ by

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu} \tag{1.19}$$

so that the power sum symmetric functions form an orthogonal basis for Λ. With respect to this inner product, we have

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$$

so that the Schur functions are orthonormal. This inner product is symmetric and positive definite and since $\omega(p_\lambda) = \pm p_\lambda$ we have

$$\langle \omega(p_\lambda), \omega(p_\mu) \rangle = \langle p_\lambda, p_\mu \rangle$$

so that ω is an isometry.

For any symmetric function $f \in \Lambda$ we let $f^\perp$ be the adjoint of multiplication by f as a linear operator on Λ (i.e., so that $\langle f^\perp g, h \rangle = \langle g, fh \rangle$ for $f, g, h \in \Lambda$). In the case of the power sum symmetric function p_n we see that

$$\begin{aligned} \langle p_n^\perp p_\lambda, p_\mu \rangle &= \langle p_\lambda, p_n p_\mu \rangle \\ &= \begin{cases} z_\lambda & \text{if } \lambda = \mu \cup \{n\} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence $p_n^\perp p_\lambda = z_\lambda z_\mu^{-1} p_\mu$ if n is a part of λ and μ is obtained from λ by removing a part of size n. Since $z_\lambda z_\mu^{-1} = n m_n(\lambda)$ where $m_n(\lambda)$ is the number of parts of size n in λ we see that

$$p_n^\perp = n \frac{\partial}{\partial p_n}.$$

Since each $f \in \Lambda$ can be written as a polynomial in terms of power sums, $f = \phi(p_1, p_2, \cdots)$, we have

$$f^\perp = \phi\left(\frac{\partial}{\partial p_1}, \frac{2\partial}{\partial p_2}, \frac{3\partial}{\partial p_3}, \cdots \right).$$

We now turn to the problem of writing the Schur and power sum symmetric functions in terms of one another. In order to do this we will require a few basic facts about the symmetric group. Complete details can be found in [22] or [31]. A permutation $\sigma \in \mathcal{S}_n$ has *cycle type* λ, where λ is a partition of n, if σ is comprised of distinct cycles of length $\lambda_1, \lambda_2, \cdots$. We denote this by $\mathrm{cyc}(\sigma) = \lambda$. The *conjugacy* *classes* in $\mathcal{S}_n$ are indexed by partitions of n so that the conjugacy class $\mathcal{C}_\lambda$, where λ is a partition of n, corresponds to the set of permutations $\sigma \in \mathcal{S}_n$ with $\mathrm{cyc}(\sigma) = \lambda$.

It is well known that the *irreducible characters* of $\mathcal{S}_n$ are indexed by partitions of n. If λ and ρ are partitions of n, then we use χ^λ_ρ to denote the evaluation of the irreducible character indexed by λ at a permutation with cycle type ρ. It is also well known that

$$\chi^\lambda_\rho = \langle s_\lambda, p_\rho \rangle.$$

This allows us to write the Schur functions and the power sum symmetric functions in terms of one another:

$$\begin{aligned} s_\lambda &= \sum_\rho z_\rho^{-1} \chi^\lambda_\rho p_\rho, \\ p_\rho &= \sum_\lambda \chi^\lambda_\rho s_\lambda. \end{aligned}$$

Example 1.3.3. *We list a number of examples of Schur functions written in terms of power sum symmetric functions which will be useful later.*

$$\begin{aligned}
s_\epsilon &= 1, \\
s_{(1)} &= p_1, \\
s_{(1^2)} &= \frac{1}{2}p_1^2 - \frac{1}{2}p_2, \\
s_{(2)} &= \frac{1}{2}p_1^2 + \frac{1}{2}p_2, \\
s_{(1^3)} &= \frac{1}{6}p_1^3 - \frac{1}{2}p_1p_2 + \frac{1}{3}p_3, \\
s_{(2,1)} &= \frac{1}{3}p_1^3 - \frac{1}{3}p_3, \\
s_{(3)} &= \frac{1}{6}p_1^3 + \frac{1}{2}p_1p_2 + \frac{1}{3}p_3, \\
s_{(1^4)} &= \frac{1}{24}p_1^4 - \frac{1}{4}p_1^2p_2 + \frac{1}{3}p_1p_3 + \frac{1}{8}p_2^2 - \frac{1}{4}p_4, \\
s_{(21^2)} &= \frac{1}{8}p_1^4 - \frac{1}{4}p_1^2p_2 - \frac{1}{8}p_2^2 + \frac{1}{4}p_4, \\
s_{(2^2)} &= \frac{1}{12}p_1^4 + \frac{1}{4}p_2^2 - \frac{1}{3}p_1p_3, \\
s_{(31)} &= \frac{1}{8}p_1^4 + \frac{1}{4}p_1^2p_2 - \frac{1}{8}p_2^2 - \frac{1}{4}p_4, \\
s_{(4)} &= \frac{1}{24}p_1^4 + \frac{1}{4}p_1^2p_2 + \frac{1}{3}p_1p_3 + \frac{1}{8}p_2^2 + \frac{1}{4}p_4.
\end{aligned}$$

The final fact that we will need concerning symmetric functions is Pieri's formula, which says that

$$h_r s_\mu = \sum_\lambda s_\lambda, \tag{1.20}$$

where the sum is over all partitions λ such that λ/μ is a horizontal r-strip. Using ω we get

$$e_r s_\mu = \sum_\lambda s_\lambda \tag{1.21}$$

where the sum is over all partitions λ such that λ/μ is a vertical r-strip. By taking adjoints we see that

$$h_r^\perp s_\mu = \sum_\lambda s_\lambda \tag{1.22}$$

where the sum is over partitions λ such that μ/λ is a horizontal r-strip and

$$e_r^\perp s_\mu = \sum_\lambda s_\lambda \tag{1.23}$$

where the sum is over partitions λ such that μ/λ is a vertical r-strip.

1.4 Thesis Outline

The remainder of this thesis is organized as follows: in Chapter 2 we construct the fermionic Fock space $\mathcal{F}$.We describe the basis of semi-infinite monomials in $\mathcal{F}$ and the related combinatorial objects called Maya diagrams. We then introduce the Lie algebra $g\ell_\infty$ of linear operators, which can be written as infinite matrices that have only a finite number of non-zero entries. We also look at the Lie group GL_∞ corresponding to $g\ell_\infty$. In preparation for chapter three we construct the algebra $\overline{a_\infty}$ which comprises infinite matrices with a finite number of non-zero diagonals and relate the action of $\overline{a_\infty}$ on $\mathcal{F}$ with the wedging and contracting operators on $\mathcal{F}$. Lastly, we find an equation which characterizes the GL_∞ orbit of a distinguished element, $\psi_0 \in \mathcal{F}$. This orbit will later become the set of solutions to the KP hierarchy and the equations which define it will become the family of partial differential equations.

In Chapter 3 we begin by discussing representations of the oscillator algebra. We show that under mild hypotheses there is a unique such representation, up to isomorphism, which can be described in terms of differential and multiplicative operators on the ring of polynomials. We use this to show that a subalgebra of $\overline{a_\infty}$ acting on $\mathcal{F}$ is in fact a representation of the oscillator algebra. Using the wedging and contracting operator description of $\overline{a_\infty}$ on $\mathcal{F}$ we then show that there is an isomorphism of representations between $\mathcal{F}$ and a direct sum of a countable number of copies of the ring of polynomials. We use this isomorphism to describe the equation characterizing one of the orbits of GL_∞ on $\mathcal{F}$ as a family of differential equations and to describe the orbit itself as a family of functions. This family of differential equations is called the KP hierarchy and the functions in the orbit are the solutions to the hierarchy. We then finish off the chapter by writing out some of the differential equations in the hierarchy.

In Chapter 4 we describe a symmetric function approach to the KP hierarchy. We do this by beginning with the family of differential equations constructed in Chapter 3 which form the KP hierarchy. We show that this family of differential equations can be written in terms of some combinatorial operators on symmetric functions. We describe explicitly the action of these combinatorial operators on Schur functions and use this to prove the well known result that solutions of the KP hierarchy are exactly those whose coefficients, when expanded in terms of Schur functions, satisfy the Plücker relations.

The final chapter is a brief account of some applications of the KP hierarchy to combinatorial problems. We begin by describing the Hurwitz problem of enumerating ramified coverings of the sphere. This leads naturally to the combinatorially defined differential operator known as the Join-Cut operator. We show that the Join-Cut operator acts as an element of GL_∞ so that the generating function for Hurwitz numbers is naturally a solution of the KP hierarchy. We then discuss a related problem of Hodge integrals over the space of genus g curves with n marked points. We show how the ELSV formula, which relates the Hurwitz numbers to Hodge integrals, can be used to prove Witten's conjecture concerning the generating function for Hodge integrals. Lastly, we show how the generating function for the related problem of double Hurwitz numbers is also a solution of the KP hierarchy.

Chapter 2

Infinite Matrix Modules and the Fermionic Fock Space

In this chapter we develop the fundamental objects involved in the representation theoretic construction of the KP hierarchy and its solutions. To this end we introduce a series of Lie algebras including and related to the infinite dimensional analogue of the finite dimensional matrix algebras. We then describe the action of these algebras on a subspace of the infinite wedge space. In doing so we also describe the action of the infinite dimensional analogue of the general linear group, GL_∞, and we find a bilinear equation which characterizes one of the orbits of GL_∞. In the following chapter we will see how this bilinear equation can be interpreted as a family of partial differential equations which comprise the KP hierarchy. The elements in the orbit then correspond to the solutions of the KP hierarchy. For the most part we follow [14] and a full description of the results which follow along with many others can be found there (see also [15] and [23]).

2.1 Fermionic Fock Space

Let

$$V = \bigoplus_{j\in\mathbb{Z}} \mathbb{C}v_j = \mathbb{C}^\infty$$

be an infinite dimensional vector space over $\mathbb{C}$ with a fixed basis $\{v_j\}_{j\in\mathbb{Z}}$.We denote by $\wedge$ the exterior (or wedge) product which is an associative antisymmetric product, so that $v_i \wedge v_j = -v_j \wedge v_i$.

We will mostly concern ourselves with a space constructed from V which is sometimes called the *(fermionic) Fock space* and is denoted by F. This subspace is constructed by taking formal linear combinations of one-way infinite exterior products of elements from V that, using antisymmetry and associativity of the wedge product, can be written in the form

$$\psi = v_{i_0} \wedge v_{i_{-1}} \wedge v_{i_{-2}} \wedge \cdots$$

where

(i) $i_0 > i_{-1} > i_{-2} > \cdots$,

(ii) $i_k = k + m$ for some integer m and k sufficiently small.

These one-way infinite wedge products are called semi-infinite monomials (or simply monomials if no confusion will arise) and the integer m is called the charge of the semi-infinite monomial ψ.

From the description of the basis for F above, we can write the charge decomposition

$$F = \bigoplus_{m \in \mathbb{Z}} F^{(m)}$$

where $F^{(m)}$ is spanned by the semi-infinite monomials in F with charge m. The subspaces $F^{(m)}$ are called the *charge m (fermionic) Fock spaces*. We will primarily be interested in the charge zero Fock space, $F^{(0)}$, although for now we will continue to use the full generality.

Each charge m Fock space has a distinguished semi-infinite monomial, denoted ψ_m, which is minimal with respect to condition (ii). By this we mean that

$$\psi_m = v_{i_0} \wedge v_{i_{-1}} \wedge v_{i_{-2}} \wedge \cdots$$

where $i_k = m+k$. These distinguished semi-infinite monomials are called the charge *m vacuum vectors*. By condition (ii) above, every monomial in $F^{(m)}$ differs in at most a finite number of places from ψ_m.

We can view any semi-infinite monomial $\psi = v_{i_0} \wedge v_{i_{-1}} \wedge \cdots$ as a two colouring of the integers. We say that, with respect to a semi-infinite monomial ψ, an integer n is black if v_n appears in ψ and that it is white if v_n does not appear in ψ. We call such a colouring a *Maya diagram* and often refer to the coloured integers as stones. For example, if n is white then we say that there is a white stone at n.

Example 2.1.1. *The Maya diagram representing the monomial $\psi = v_3 \wedge v_2 \wedge v_0 \wedge v_{-2} \wedge v_{-3} \wedge v_{-5} \wedge \cdots$ is shown in Figure 2.1.*

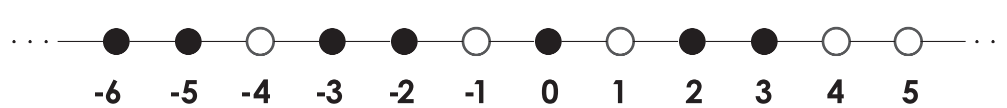

Figure 2.1: Maya Diagram Example

Since we know that every monomial in $F^{(m)}$ differs from ψ_m in at most a finite number of places, this implies that the Maya diagram corresponding to a monomial in $F^{(m)}$ can only differ from the Maya diagram corresponding to ψ_m by a finite number of exchanges of black and white stones. By this we mean that one starts with the Maya diagram corresponding to ψ_m, and then chooses a white stone and a black stone and swaps the two colours for each exchange.

We may view Maya diagrams as paths in the lower right quadrant of the integer lattice where we interpret black stones as denoting 'up' steps and white stones as denoting 'right' steps. Using this notation we see that ψ_m represents the portion of the y-axis below the point m on the x-axis and the portion of the x-axis to the right of the y-axis.

Example 2.1.2. *If we use the monomial from example 2.1.1 then we get the diagram in Figure 2.2.*

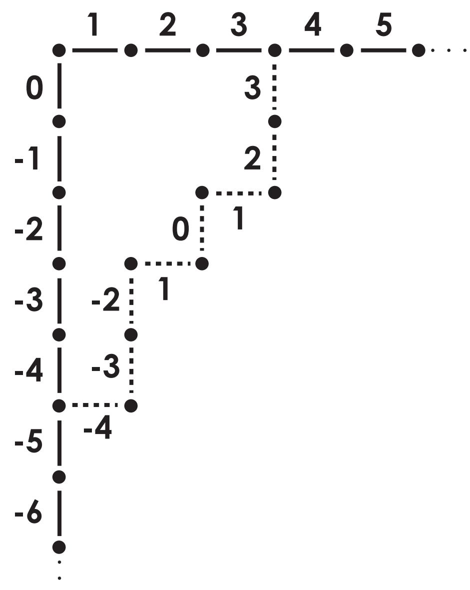

Figure 2.2: Semi-Infinite Monomial Example

In Figure 2.2 the solid black line represents ψ_0 and the dashed line represents the monomial from the previous example. Also, the steps in the diagram have been labeled so that the labels corresponding to up steps appear to the left and the labels corresponding to right steps have been placed below. Note that this configuration outlines the Young diagram of the partition $3^2 21^2$.

In general, if we construct a diagram as in Example 2.1.2 by first drawing the path corresponding to ψ_m and superimposing the path corresponding to a semi-infinite monomial $\psi \in F^{(m)}$ then it is easy to see that the region between the two paths corresponds to the Young diagram of a partition. If we have some basis element ψ in $F^{(m)}$ then we can write it as

$$\psi = v_{i_0} \wedge v_{i_{-1}} \wedge \cdots \wedge v_{i_{-n+1}} \wedge v_{m-n} \wedge v_{m-n-1} \wedge \cdots$$

where $i_{k+1} < i_k$ and we can relate it to the partition λ formed above using the formula

$$\lambda_{k+1} = i_{-k} + k - m$$

for $k \geq 0$. It is easy to see that this formula gives us the same partition as was described above since this measures the number of right steps that occur before the i-th up step, which is the size of the i-th part of the associated partition.

We sometimes refer to the semi-infinite monomial ψ associated to the partition λ as v_λ, where

$$v_\lambda = v_{i_0} \wedge v_{i_{-1}} \wedge v_{i_{-n+1}} \wedge v_{m-n} \wedge v_{m-n-1} \wedge \cdots ,$$

with $i_{-k} = \lambda_{k+1} - k + m$. Note that there is some ambiguity in the notation for v_λ; however this should not be an issue as the charge m of a semi-infinite monomial will always be clear from the context. From this we see that if we define $\deg(v_\lambda) = |\lambda|$ then for $v_\lambda \in F^{(m)}$

$$\deg(v_\lambda) = \sum_{k=0}^{\infty} (i_{-k} + k - m).$$

This gives us the decomposition

$$F^{(m)} = \bigoplus_{k \geq 0} F_k^{(m)}$$

where $F_k^{(m)}$ is spanned by all semi-infinite monomials with charge m and degree k. This decomposition is sometimes called the *energy decomposition* and the degree is called *energy*. One last thing to note is that the comments above imply the dimension of the subspace $F_k^{(m)}$ is equal to the number of partitions of k.

2.2 The algebras $g\ell_\infty$ and $\overline{a_\infty}$

We now describe two related associative algebras $g\ell_\infty$ and $\overline{a_\infty}$. In the next section we construct representations of $g\ell_\infty$ and $\overline{a_\infty}$ on F together with the representation of a central extension of $\overline{a_\infty}$.

Since our underlying vector space V is $\mathbb{C}^\infty$ it is natural to consider the Lie algebra analogous to the algebra of matrices on a finite dimensional vector space. To that end we define

$$g\ell_\infty = \{(a_{i,j})_{i,j \in \mathbb{Z}} | \text{ all but a finite number of } a_{i,j} = 0\}.$$

We view $g\ell_\infty$ as a Lie algebra with juxtaposition corresponding to standard matrix multiplication,

$$(ab)_{i,j} = \sum_{k \in \mathbb{Z}} a_{i,k} b_{k,j},$$

for $a, b \in g\ell_\infty$, and the standard commutator bracket, $[a, b] = ab - ba$.

Most of our computations will be done using a convenient linear basis of $g\ell_\infty$. Let $E_{i,j}$ be the element of $g\ell_\infty$ with a 1 in its (i,j) entry and 0 elsewhere. It is clear that this forms a linear basis of $g\ell_\infty$.

By construction we see that

$$E_{i,j}v_k = \delta_{j,k}v_i,$$

$$E_{i,j}E_{m,n} = \delta_{j,m}E_{i,n}.$$

From this we see that

$$[E_{i,j}, E_{m,n}] = \delta_{j,m}E_{i,n} - \delta_{n,i}E_{m,j},$$

from which we can construct all other Lie brackets (by linearity).

Since $g\ell_\infty$ is a Lie algebra we can construct its associated group GL_∞. This is defined as

$$GL_\infty = \{A = (a_{i,j})_{i,j\in\mathbb{Z}} | A \text{ is invertible and all but a finite number of } a_{i,j} - \delta_{i,j} = 0\},$$

where we take matrix multiplication as our group operation. In a moment we will introduce the shift operators which will be important in the next chapter. Since the shift operators are not members of $g\ell_\infty$ we will need to define a larger Lie algebra of which $g\ell_\infty$ is a Lie subalgebra. We define the algebra

$$\overline{a_\infty} = \{(a_{i,j})_{i,j\in\mathbb{Z}} | a_{i,j} = 0 \text{ for } |i-j| \gg 0\}.$$

In other words, the infinite matrices in $\overline{a_\infty}$ are those which have a finite number of non-zero diagonals. It is easy to check that matrix products are well defined in $\overline{a_\infty}$, and that $g\ell_\infty$ is a Lie subalgebra of $\overline{a_\infty}$.

Since we will need them later, we introduce the *shift operators* Λ_k. These are defined as

$$\Lambda_k v_j = v_{j-k}.$$

From the description of the matrices E_{ij} and their action on v_j we see immediately that

$$\Lambda_k = \sum_{i\in\mathbb{Z}} E_{i,i+k}.$$

In other words, Λ_k is the matrix with 1 in each entry of the k-th diagonal and 0 elsewhere (here we view the principal diagonal as the 0-th diagonal and positive diagonals are above the 0-th diagonal, etc.).

From the definition of Λ_k we see that

$$\begin{aligned}
[\Lambda_j, \Lambda_k] &= \sum_{m,n\in\mathbb{Z}} [E_{m,m+j}, E_{n,n+k}] \\
&= \sum_{m,n\in\mathbb{Z}} (\delta_{m+j,j}E_{m,n+k} - \delta_{n+k,m}E_{n,m+j}) \\
&= 0
\end{aligned}$$

and so the Λ_k form a commutative subalgebra of $\overline{a_\infty}$.

2.3 Representations of GL_∞ and $g\ell_\infty$ in F

Using the action of $g\ell_\infty$ and GL_∞ on V described in the previous section we can construct representations R of GL_∞ and r of $g\ell_\infty$ on F by

$$R(A)(v_{i_1} \wedge v_{i_2} \wedge \cdots) = Av_{i_1} \wedge Av_{i_2} \wedge \cdots \tag{2.1}$$

and

$$r(a)(v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots) = av_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots + v_{i_1} \wedge av_{i_2} \wedge v_{i_3} \wedge \cdots + \cdots . \tag{2.2}$$

Note that the sum above is finite since a has only a finite number of non-zero entries (by definition), hence $av_i = 0$ for all but finitely many v_i.

Also, the representations r and R are related by

$$\begin{aligned} R(\exp(a))v &= \exp(a)v_{i_1} \wedge \exp(a)v_{i_2} \wedge \cdots \\ &= \left(1 + a + \frac{a^2}{2} + \cdots\right) v_{i_1} \wedge \left(1 + a + \frac{a^2}{2}\right) v_{i_2} \wedge \cdots \\ &= v + (av_{i_1} \wedge v_{i_2} \wedge \cdots + v_{i_1} \wedge av_{i_2} \wedge v_{i_3} \wedge \cdots +) + \cdots \\ &= v + r(a)v + \frac{r^2(a)}{2} v + \cdots \\ &= \exp(r(a))v, \end{aligned}$$

for $a \in g\ell_\infty$ and any semi-infinite monomial $v \in F$.

We see that

$$r(E_{i,j})v_{i_1} \wedge v_{i_2} \wedge \cdots = \begin{cases} 0 \text{ if } j \notin \{i_1, i_2, \cdots\}, \\ v_{i_1} \wedge \cdots \wedge v_{i_{k-1}} \wedge v_i \wedge v_{i_{k+1}} \wedge \cdots \text{ if } j = i_k \end{cases} .$$

It is clear that $r(E_{i,j})v_{i_1} \wedge v_{i_2} \wedge \cdots = 0$ if $i \in \{i_1, i_2, \cdots\}$ as well, using the anti-symmetric property of the wedge product. It is also clear that $r(E_{i,j})$ maps $F^{(m)}$ to itself for each m since its action is to replace one basis element in a semi-infinite monomial with another.

In terms of Maya diagrams, the action of $r(E_{i,j})$ on some semi-infinite monomial ψ is as follows: If j is black and i is white then switch the colours on i and j and multiply by $(-1)^n$ where n is the number of black stones between i and j. Otherwise the result is zero. Using this description it is again clear that $r(E_{i,j})$ does not change the charge of a monomial.

Since each basis element $\psi \in F^{(m)}$ can be written as

$$\psi = v_{i_m} \wedge \cdots \wedge v_{i_{m-k}} \wedge v_{m-k-1} \wedge \cdots$$

we can in fact write it as

$$\psi = r(E_{i_m,m}) \cdots r(E_{i_{m-k},m-k})\psi_m. \tag{2.3}$$

We think of $\mathcal{F}^{(m)}$ as being generated by ψ_m since (2.3) allows us to describe any charge m semi-infinite monomial as an element of the $g\ell_\infty$ orbit of ψ_m.

Definition 2.3.1. *We say that* ω *is an* anti-linear anti-involution *if it is an* $\mathbb{R}$*-linear involution on a complex Lie algebra* $\mathcal{A}$ *such that*

$$\omega(\lambda x) = \overline{\lambda}\omega(x)$$

for $\lambda \in \mathbb{C}$ *and* $x \in \mathcal{A}$ *and such that*

$$\omega([x, y]) = [\omega(y), \omega(x)]$$

for $x, y \in \mathcal{A}$.

Definition 2.3.2. *Let* V *be a representation space of a complex Lie algebra* $\mathcal{A}$ *and let* $\langle\cdot|\cdot\rangle$ *be a Hermitian form on* V *with an anti-linear anti-involution* ω. *We say that* $\langle\cdot|\cdot\rangle$ *is* contravariant *if*

$$\langle x(u)|v\rangle = \langle u|\omega(x)(v)\rangle$$

holds for all $x \in \mathcal{A}$ *and* $u, v \in V$.

If the Hermitian form in Definition 2.3.2 is non-degenerate then for any $x \in \mathcal{A}$ there exists a Hermitian adjoint x^* such that $\langle x(u)|v\rangle = \langle u|x^*(v)\rangle$ for all $u, v \in V$. This implies that the anti-linear anti-involution ω must in fact be the operator that takes x to x^*. In what follows we will often have a non-degenerate Hermitian form and when this is the case we use x^* to denote $\omega(x)$ as it is more descriptive.

Definition 2.3.3. *We further say that the representation is* unitary *if the Hermitian form is contravariant and*

$$\langle v|v\rangle > 0 \textit{ for all } v \in V, v \neq 0.$$

We may define a Hermitian form $\langle\cdot|\cdot\rangle$ on F by requiring that semi-infinite monomials determine an orthonormal basis for F. If we let a^* denote the transposed conjugate of a matrix a then we can see very quickly (by considering the basis $E_{i,j}$) that for any two semi-infinite monomials ψ, ψ',

$$\langle r(a)\psi|\psi'\rangle = \langle\psi|r(a^*)\psi'\rangle.$$

This tells us that $\langle\cdot|\cdot\rangle$ is contravariant and that the representation r of $g\ell_\infty$ on F is unitary. In fact, we can say more.

Proposition 2.3.4. *The representation* r *of* $g\ell_\infty$ *on* F *is a direct sum of irreducible unitary representations* r_m, *where* r_m *is the restriction of* r *to* $F^{(m)}$.

Proof. The fact that r is a direct sum of unitary representations r_m in $F^{(m)}$ follows from the fact that the charge decomposition is orthogonal with respect to the Hermitian form $\langle\cdot|\cdot\rangle$.

To see that each r_m is irreducible note that we have an energy decomposition of each charge m Fock space as

$$F^{(m)} = \sum_{k \geq 0} F_k^{(m)}$$

and that this decomposition is also orthogonal.

Suppose $U \subset F^{(m)}$ is an invariant subspace with respect to r_m. Then U must also respect the energy decomposition (for details as to why this is true, see Lemma 1.1 in [14]),

$$U = \sum_{k \geq 0} (U \cap F_k^{(m)})$$

and so must its orthogonal complement $U^{\perp}$. However, since ψ_m spans all of $F_0^{(m)}$ it must be contained in only one of U or $U^{\perp}$ and we may assume without loss of generality that it is contained in U.

From our comment earlier that each basis element in $F^{(m)}$ can be written as the image of ψ_m under $g\ell_\infty$ we see that in fact each of the fixed energy subspaces of $F^{(m)}$ must also be contained in U so that $U^{\perp} = 0$. Since this argument is independent of m we see that each r_m must in fact be irreducible. $\square$

Lastly, we record a formula for the action of $R(A), A \in GL_\infty$ on $F^{(m)}$ which will be denoted $R_m(A)$. This formula can be proved easily using the standard calculus of exterior algebra.

We have that

$$R_m(A)(v_{i_m} \wedge v_{i_{m-1}} \wedge \cdots) = \sum_{j_m > j_{m-1} > \cdots} \left(\det A^{i_m, i_{m-1}, \cdots}_{j_m, j_{m-1}, \cdots}\right) v_{j_m} \wedge v_{j_{m-1}} \wedge \cdots, \quad (2.4)$$

where $A^{i_m, i_{m-1}, \cdots}_{j_m, j_{m-1}, \cdots}$ denotes the sub-matrix of the infinite matrix A with rows indexed by j_m, j_{m-1}, $\cdots$ and columns indexed by $i_m, i_{m-1}, \cdots$. Here we view the determinant of an infinite matrix as the product of its eigenvalues. This is well defined since, by definition, each matrix in question is block diagonalizable consisting of a finite matrix and an infinite identity matrix.

2.4 Representations of a_∞ in F

Let $a_k \in \overline{a_\infty}$ represent an infinite matrix of the form

$$a_k = \sum_{j \in \mathbb{Z}} \lambda_j E_{j,j+k},$$

so that a_k has non-zero entries ($\lambda_j \in \mathbb{C}$) only on its k-th diagonal. Then each matrix in $\overline{a_\infty}$ can be written as a finite sum of matrices of this form.

Since we would like to construct a representation of $\overline{a_\infty}$ on F similar to the $g\ell_\infty$ representation r that we discussed in the previous section, it would seem natural to simply apply (2.2) for r to $\overline{a_\infty}$.

For $k \neq 0$ we see that $r(a_k)\psi_m$ is a finite linear combination of semi-infinite monomials in $F^{(m)}$. Since $r(E_{i,i+k})\psi_m = 0$ when $i + k > m$ and when $i \leq m$. However, for $k = 0$ we see that

$$r(a_0)\psi_m = (\lambda_m + \lambda_{m-1} + \cdots)\psi_m,$$

which is potentially a divergent sum. Since each semi-infinite monomial in $F^{(m)}$ is generated by ψ_m via (2.3), the representation r is not well defined on $\overline{a_\infty}$. In order to fix this problem it will be easier to work with a slightly larger algebra than to work with $\overline{a_\infty}$ directly.

First, we define the 2-cocycle α on $\overline{a_\infty}$ by

$$\alpha(E_{i,j}, E_{m,n}) = \begin{cases} 1 \text{ if } n = i \leq 0 \text{ and } m = j \geq 1, \\ -1 \text{ if } n = i \geq 1 \text{ and } m = j \leq 0, \\ 0 \text{ otherwise} \end{cases}$$

and extend linearly.

Now, consider the central extension

$$a_\infty = \overline{a_\infty} \oplus \mathbb{C}c$$

where c is some new central element. We define the Lie bracket on a_∞ by

$$[a, b] = ab - ba + \alpha(a, b)c$$

where $a, b \in \overline{a_\infty}$. We then extend linearly, keeping in mind that c is central. It is easy to see that this is a well defined Lie algebra and that $\overline{a_\infty} = a_\infty/\mathbb{C}c$.

We now define a representation of a_∞ on $F^{(m)}$ which we denote by $\hat{r}_m$. We also use $\hat{r}$ to denote the direct sum of representations $\hat{r}_m$ on F, as we do with r and r_m. Let

$$\hat{r}_m(E_{i,j}) = \begin{cases} r_m(E_{i,j}) \text{ if } i \neq j \text{ or } i = j > 0, \\ r_m(E_{i,j}) - I \text{ if } i = j \leq 0. \end{cases},$$

and

$$\hat{r}_m(c) = 1.$$

Proposition 2.4.1. *The map $\hat{r}_m$ does in fact give a representation of a_∞ on $F^{(m)}$. Furthermore, if we declare that $c^* = c$ then $\hat{r}_m$ is unitary and irreducible.*

Proof. To see that $\hat{r}_m$ gives a representation of a_∞ on $F^{(m)}$ we need only check that it satisfies the Lie bracket. In fact, it suffices to check that $\hat{r}_m$ respects the Lie bracket on the elementary matrices $E_{i,j}$. To make this easier to see, we can rewrite the commutation relations of the $E_{i,j}$ as a set of four relations as follows:

(i) $[E_{i,j}, E_{k,l}] = 0$ for $j \neq k, l \neq i$,

(ii) $[E_{i,j}, E_{j,l}] = E_{i,l}$ for $l \neq i$,

(iii) $[E_{i,j}, E_{k,i}] = -E_{k,j}$ for $j \neq k$,

(iv) $[E_{i,j}, E_{j,i}] = E_{i,i} - E_{j,j} + \alpha(E_{i,j}, E_{j,i})c$.

For relations $(i), (ii)$ and (iii) we see that

$$\begin{aligned}
[\hat{r}_m(E_{i,j}), \hat{r}_m(E_{k,l})] &= [r_m(E_{i,j}, r_m(E_{k,l})] \text{ for } j \neq k, l \neq i, \\
[\hat{r}_m(E_{i,j}), \hat{r}_m(E_{j,l})] &= [r_m(E_{i,j}, r_m(E_{j,l})] \text{ for } l \neq i, \\
[\hat{r}_m(E_{i,j}), \hat{r}_m(E_{k,i})] &= [r_m(E_{i,j}, r_m(E_{k,i})] \text{ for } j \neq k,
\end{aligned}$$

since I commutes with everything. Also,

$$\begin{aligned}
\hat{r}_m(0) &= r_m(0), \\
\hat{r}_m(E_{i,l}) &= r_m(E_{i,l}) \text{ if } l \neq i \text{ and} \\
\hat{r}_m(-E_{k,j}) &= r_m(-E_{k,j}) \text{ if } j \neq k.
\end{aligned}$$

Since r_m respects the Lie bracket on $g\ell_\infty$, we see that $\hat{r}_m$ respects relations (i), (ii) and (iii).

For relation (iv) we have

$$[\hat{r}_m(E_{i,j}), \hat{r}_m(E_{j,i})] = \hat{r}_m(E_{i,i}) - \hat{r}_m(E_{j,j}) + \alpha(E_{i,j}, E_{j,i})I$$

and so we see immediately that $\hat{r}_m$ respects relation (iv).

It is easy to see that when $c^* = c$, $\hat{r}_m$ is unitary and irreducibility follows in much the same way as in Proposition 2.3.4. □

Since a_∞ is a central extension of $\overline{a_\infty}$, $\hat{r}_m$ is in fact a projective representation of $\overline{a_\infty}$ on $F^{(m)}$ arising from the 2-cocycle α.

2.5 Wedging and Contracting

Recall that

$$V = \bigoplus_{j \in \mathbb{Z}} \mathbb{C} v_j.$$

For each v_j we can construct the linear functional v_j^* on V by

$$v_j^*(v_i) = \delta_{i,j}$$

for $i, j \in \mathbb{Z}$. We then construct the restricted dual of V as

$$V^* = \bigoplus_{j \in \mathbb{Z}} \mathbb{C} v_j^*.$$

Vectors in V and V^* define operators on F as follows. Each $v \in V$ defines a *wedging operator* $\hat{v}$ on F by

$$\hat{v}(v_{i_1} \wedge v_{i_2} \wedge \cdots) = v \wedge v_{i_1} \wedge v_{i_2} \wedge \cdots .$$

Each $f \in V^*$ defines a *contraction operator* $\check{f}$ on F by

$$\check{f}(v_{i_1} \wedge v_{i_2} \wedge \cdots) = \sum_{k \geq 1} (-1)^{k+1} f(v_{i_k}) v_{i_1} \wedge \cdots \wedge v_{i_{k-1}} \wedge v_{i_{k+1}} \wedge \cdots .$$

When describing the actions of wedging and contraction it suffices to look at only the basis elements and their duals. In doing so we can construct a combinatorial rule for the action of wedging and contraction on Maya diagrams.

For the wedging operator $\hat{v}_j$, we see that

$$\begin{aligned} \hat{v}_j(v_{i_1} \wedge v_{i_2} \wedge \cdots) &= v_j \wedge v_{i_1} \wedge v_{i_2} \wedge \cdots \\ &= (-1)^k v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge v_j \wedge v_{i_{k+1}} \wedge \cdots \end{aligned}$$

if $i_k < j < i_{k+1}$ and is zero otherwise. In terms of Maya diagrams this corresponds to looking at the stone at index j. If this stone is white then we change it to black and multiply by $(-1)^k$ where k is the number of black stones to the left of j and the result vanishes otherwise.

Example 2.5.1. *If we start with the semi-infinite monomial* $\psi = v_3 \wedge v_2 \wedge v_0 \wedge v_{-2} \wedge v_{-3} \wedge v_{-5} \wedge \cdots$ *from the earlier examples then we see that*

$$\varphi = \hat{v}_{-1}\psi = (-1)^3 v_3 \wedge v_2 \wedge v_0 \wedge v_{-1} \wedge v_{-2} \wedge v_{-3} \wedge v_{-5} \wedge \cdots .$$

We can represent this operation visually as in Figure 2.3.

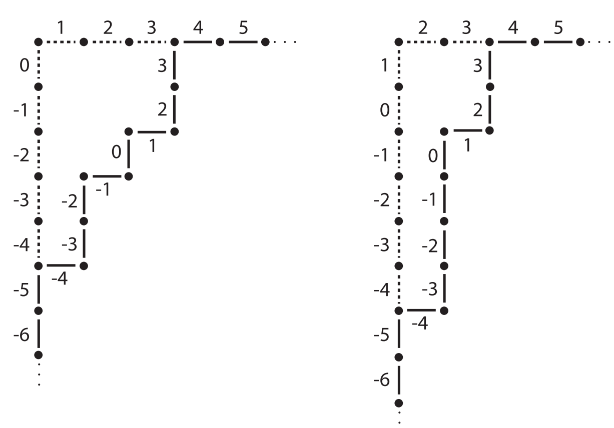

Figure 2.3: Wedging Example

In Figure 2.3 we have drawn the labeled partition corresponding to ψ *on the left and the one for* φ *on the right. This operation can be viewed as changing the* *horizontal line labeled with -1 to a vertical line and multiplying by $(-1)^k$ where k is the number of vertical lines with labels greater than -1 in ψ. Notice that the label in the upper left corner has increased from 0 to 1 which represents the fact that $\varphi \in F^{(1)}$ and $\psi \in F^{(0)}$.*

Similarly, for the contraction operator $\check{v}_j^*$, we see that

$$\begin{aligned} \check{v}_j^*(v_{i_1} \wedge v_{i_2} \wedge \cdots) &= v_j^*(v_{i_1}) v_{i_2} \wedge v_{i_3} \wedge \cdots - v_j^*(v_{i_2}) v_{i_1} \wedge v_{i_3} \wedge \cdots + \cdots \\ &= (-1)^k v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge v_{i_{k+2}} \wedge \cdots \end{aligned}$$

where $i_{k+1} = j$ and is zero otherwise. In this case the interpretation on Maya diagrams is that we look at index j and if it contains a black stone then we change it into a white stone and multiply by $(-1)^k$ where k is the number of black stones to the left of index j. If j does not contain a white stone then the result vanishes.

Example 2.5.2. *If we start with the semi-infinite monomial $\psi = v_3 \wedge v_2 \wedge v_0 \wedge v_{-2} \wedge v_{-3} \wedge v_{-5} \wedge \cdots$ from the earlier examples then we see that*

$$\varphi = \check{v}_0^* \psi = (-1)^2 v_3 \wedge v_2 \wedge v_{-2} \wedge v_{-3} \wedge v_{-5} \wedge \cdots .$$

We can represent this operation visually as in Figure 2.4.

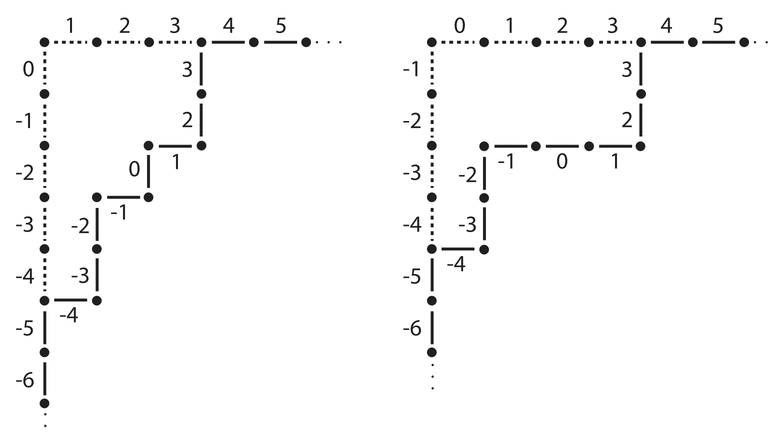

Figure 2.4: Contracting Example

In Figure 2.4 we have drawn the labeled partition corresponding to ψ on the left and the one for φ on the right. Similar to example 2.5.1, this operation can be viewed as changing the vertical line labeled with 0 to a horizontal line and multiplying by $(-1)^k$ where k is the number of vertical lines with labels greater than 0 in ψ. Notice that in this case the label in the upper left corner has decreased from 0 to -1 which represents the fact that $\varphi \in F^{(-1)}$ and $\psi \in F^{(0)}$.

From the combinatorial description of $\hat{v}_j$ and $\check{v}_j^*$ (or from their algebraic descriptions) we see that they are adjoint with respect to the Hermitian form $\langle \cdot | \cdot \rangle$. Also, any wedging operator maps $F^{(m)}$ to $F^{(m+1)}$ and any contraction operator maps $F^{(m+1)}$ to $F^{(m)}$. We also see immediately that

$$\hat{r}(E_{i,j}) = \begin{cases} \hat{v}_i\check{v}_j^* \text{ if } i \neq j \text{ or } i = j > 0, \\ \hat{v}_i\check{v}_j^* - I \text{ if } i = j \leq 0. \end{cases}$$

Thus

$$\hat{r}(\Lambda_k) = \sum_{i\in\mathbb{Z}} \hat{v}_i\check{v}_{i+k}^* \text{ for } k \neq 0,$$

$$\hat{r}(\Lambda_0) = \sum_{i>0} \hat{v}_i\check{v}_i^* - \sum_{i\leq 0} \check{v}_i^*\hat{v}_i.$$

The expression for $\hat{r}(\Lambda_0)$ follows from the fact that $\hat{v}_i\check{v}_i^* - I = -\check{v}_i^*\hat{v}_i$, which can be seen immediately from the combinatorial description of the wedging and contracting operators. In fact, this is just one of a family of relations that the wedging and contracting operators satisfy.

Proposition 2.5.3. *The algebra of wedging and contracting operators satisfy the following anti-commutation relations (here $[a,b]_+ = ab+ba$ is the anti-commutator bracket).*

$$[\hat{v}_i, \hat{v}_j]_+ = 0,\ [\check{v}_i^*, \check{v}_j^*]_+ = 0,\ [\hat{v}_i, \check{v}_j^*]_+ = \delta_{i,j}. \tag{2.5}$$

Proof. Let i and j be integers such that $i < j$ and let ψ be a semi-infinite monomial.

For the first relation if either i or j are black with respect to ψ then $\hat{v}_i\hat{v}_j\psi = \hat{v}_j\hat{v}_i\psi = 0$. If both i and j are white with respect to ψ then regardless of the order in which we apply $\hat{v}_i$ and $\hat{v}_j$ we will arrive at the same Maya diagram. The only thing that will change is the associated sign. Since applying $\hat{v}_i$ increases the number of black stones to the left of stone j by one we see easily that $\hat{v}_j\hat{v}_i\psi = -\hat{v}_i\hat{v}_j\psi$ which completes the proof of the first relation.

The analysis for the second relation is almost exactly the same as the first with white stones exchanged for black stones.

For the final relation, the case when $i \neq j$ follows in the same manner as the first two relations described above. The case when $i = j$ is a little different so we will now assume that this is the case. If the stone i is white then $\hat{v}_i\check{v}_i^*\psi = 0$. Also, $\varphi = \hat{v}_i\psi$ has stone i coloured black and has sign $(-1)^k$ times that of ψ where k is the number of black stones to the left of stone i. Thus, it is easy to see that in this case $\check{v}_i^*\hat{v}_i\psi = \psi$. The case when stone i is black follows by symmetry. □

The algebra generated by the wedging and contracting operators is sometimes referred to as a Clifford algebra.

Using the commutation relations in (2.5) we see that, for $j \neq 0$,

$$[\hat{r}(\Lambda_j), \hat{v}_k] = \sum_{n\in\mathbb{Z}} \hat{v}_n\check{v}_{n+j}^*\hat{v}_k - \hat{v}_k\hat{v}_n\check{v}_{n+j}^*.$$

For $k \neq n+j$ we have

$$\hat{v}_n \check{v}^*_{n+j} \hat{v}_k = -\hat{v}_n \hat{v}_k \check{v}^*_{n+j} = \hat{v}_k \hat{v}_n \check{v}^*_{n+j}.$$

Also, for $k = n+j$ we have

$$\hat{v}_n \check{v}^*_{n+j} \hat{v}_k = -\hat{v}_n \hat{v}_k \check{v}^*_{n+j} + \hat{v}_n = \hat{v}_k \hat{v}_n \check{v}^*_{n+j} + \hat{v}_n.$$

Putting these together we have

$$[\hat{r}(\Lambda_j), \hat{v}_k] = \hat{v}_{k-j},$$

and,

$$[\hat{r}(\Lambda_j), \check{v}^*_k] = -\check{v}^*_{k+j}.$$

2.6 The orbit $GL_\infty \psi_0$

We will now describe the elements in the orbit $GL_\infty \psi_0$ which (as we shall see in the following chapter) comprise the solutions of the KP hierarchy of partial differential equations.

Theorem 2.6.1. *If $\tau \in GL_\infty \psi_0$, then τ is a solution of the equation*

$$\sum_{j \in \mathbb{Z}} \hat{v}_j(\tau) \otimes \check{v}^*_j(\tau) = 0. \tag{2.6}$$

Conversely, if $\tau \in F^{(0)}, \tau \neq 0$ and τ satisfies (2.6)*, then $\tau \in GL_\infty \psi_0$.*

Proof. $\hat{v}_j(\psi_0) = 0$ for $j \leq 0$ and $\check{v}^*_j(\psi_0) = 0$ for $j > 0$ so that

$$\sum_{j \in \mathbb{Z}} \hat{v}_j(\psi_0) \otimes \check{v}^*_j(\psi_0) = 0. \tag{2.7}$$

Suppose $\tau \in GL_\infty \psi_0$ is of the form

$$\tau = R_0(A)\psi_0,$$

where $A \in GL_\infty$.

Using the definition of the wedging and contracting operators along with the definition of the representation R_0 we see that

$$R_0(A)\hat{v}R_0(A)^{-1} = \hat{w}, \quad \text{where } w = Av, \textit{and}$$

$$R_0(A)\check{f}R_0(A)^{-1} = \check{g}, \quad \text{where } g = (A^{-1})^T f.$$

Here A^T is the transpose of A.

We will denote the entries in A and A^{-1} with respect to the basis $\{v_i\}$ by $a_{i,j}$ and $\overline{a_{i,j}}$ respectively. Explicitly, we have that

$$Av_j = \sum_i a_{j,i} v_i, \tag{2.8}$$

$$(A^{-1})^T v_j^* = \sum_k \overline{a_{k,j}} v_k^*, \tag{2.9}$$

$$\sum_j \overline{a_{k,j}} a_{j,i} = \delta_{k,i}. \tag{2.10}$$

If we apply the operator $R_0(A)$ to (2.7) and then use the fact that $\tau = R_0(A)\psi_0$ we get that

$$\sum_j R_0(A)\hat{v}_j R_0(A)^{-1}(\tau) \otimes R_0(A)\check{v}_j^* R_0(A)^{-1}(\tau) = 0.$$

Using the relations (2.8), (2.9) and (2.10) we see that this becomes

$$\sum_{i,j,k} a_{j,i}\hat{v}_i(\tau) \otimes \overline{a_{k,j}}\check{v}_k^*(\tau) = 0,$$

which we can rewrite as

$$\sum_{i,k} \left(\sum_j \overline{a_{k,j}} a_{j,i} \right) \hat{v}_i(\tau) \otimes \check{v}_k^*(\tau) = 0,$$

which is equal to (2.6) after applying (2.10).

To see the other direction, let $\tau \in F^{(0)}$, $\tau \neq 0$ and τ satisfy (2.6). We can write

$$\tau = \sum_{k=1}^{N} c_k \tau_k,$$

a linear combination with non-zero coefficients c_k of some semi-infinite monomials τ_k, such that τ_1 is one of the semi-infinite monomials with greatest charge; we may assume that $c_1 = 1$. If among the τ_i with $i > 1$ there exists a semi-infinite monomial, say τ_2, of the form

$$r_0(E_{i,j})\tau_1 \text{ with } i < j, \tag{2.11}$$

then

$$\tau' = R_0(\exp(-c_2 E_{i,j}))\tau$$

does not contain the semi-infinite monomial τ_2 and satisfies (2.6) since $\tau \in GL_\infty \psi_0$ and $R_0(\exp(-c_2 E_{i,j})) \in GL_\infty$. We can continue this procedure a finite number of times to arrive at an element of the form $\tau_1 + \phi$, where none of the semi-infinite monomials in ϕ are of the form (2.11). The finiteness follows from the fact that

$$r_0(E_{i,j})\psi_0 = 0 \text{ when } i < j$$

and (2.3).

Since $\tau_1 + \phi$ satisfies (2.6) we see that

$$0 = \sum_{j \in \mathbb{Z}} \hat{v}_j(\tau_1 + \phi) \otimes \check{v}_j^*(\tau_1 + \phi) = \sum_{j \in \mathbb{Z}} \hat{v}_j(\tau_1) \otimes \check{v}_j^*(\phi) + \sum_{j \in \mathbb{Z}} \hat{v}_j(\phi) \otimes \check{v}_j^*(\tau_1) + \sum_{j \in \mathbb{Z}} \hat{v}_j(\phi) \otimes \check{v}_j^*(\phi). \tag{2.12}$$

We may rewrite the right hand side of (2.12) as

$$\sum_{j \in \mathbb{Z}} \hat{v}_j(\tau_1) \otimes \check{v}_j^*(\phi) + \sum_{j \in \mathbb{Z}} \hat{v}_j(\phi) \otimes \check{v}_j^*(\tau_1) + \sum_{j \in \mathbb{Z}} \hat{v}_j(\phi) \otimes \check{v}_j^*(\phi) = \sum_{i \in \mathbb{Z}} \hat{v}_i(\tau_1) \otimes \Gamma_i + \sum_{j \in \mathbb{Z}} \gamma_j \otimes \widetilde{\Gamma}_j$$

where the set $\{\hat{v}_i(\tau_1), \gamma_j\}_{i,j \in \mathbb{Z}}$ is linearly independent and $\Gamma_i, \widetilde{\Gamma}_j$ are elements in F.

Suppose $\hat{v}_i(\tau_1) \neq 0$ for some $i \in \mathbb{Z}$. Since

$$\hat{v}_i(\tau_1) \otimes \Gamma_i = 0$$

we know that $\Gamma_i = 0$. If $\Gamma_i \neq \check{v}_i^*(\phi)$ then it follows from (2.12) that there must be some semi-infinite monomial φ in ϕ for which

$$\hat{v}_i(\tau_1) = \pm \hat{v}_j(\varphi).$$

Also, since φ has at most the same energy and is not equal to τ_1, we know that $i < j$. We also see that

$$\begin{aligned}
\varphi &= \check{v}_j^* \hat{v}_j(\varphi) \\
&= \pm \check{v}_j^* \hat{v}_i(\tau_1) \\
&= \mp \hat{v}_i \check{v}_j^*(\tau_1) \\
&= \mp r_0(E_{i,j}) \tau_1,
\end{aligned}$$

where $i < j$. This is a contradiction and so it must be true that $\Gamma_i = \check{v}_i^*(\phi)$. This, however, implies that $\check{v}_i^*(\phi) = 0$ since $\hat{v}_i(\tau_1) \neq 0$. Since $\hat{v}_i(\tau_1)$ is non-zero for an infinite number of positive integers, ϕ must be zero (i.e., there are no semi-infinite monomials ρ with the property that $\check{v}_i^*(\rho) = 0$ for an infinite number of positive integers).

Finally, since τ_1 is a semi-infinite monomial, by (2.3) $\tau_1 \in GL_\infty \psi_0$, and so $\tau \in GL_\infty \psi_0$.

□

Chapter 3

Realizing The $GL_\infty\psi_0$ Orbit

The purpose of this chapter is to construct a realization of the Fock space described in the previous chapter. That is, we will construct an isomorphism from F to some polynomial algebra under which the representations r, R and $\hat{r}$ map to differential operators. This will allow us to describe the $GL_\infty\psi_0$ orbit as a family of polynomials which simultaneously satisfy a family of partial differential equations. As in the previous chapter, we will mostly be following [14] and more information can be found in [15] and [23].

3.1 The Oscillator Algebra $\mathcal{A}$

Let $\mathcal{A}$ be the complex Lie algebra with basis $\{a_n, n \in \mathbb{Z}; \hbar\}$ and commutation relations

$$\begin{aligned} [\hbar, a_n] &= 0 \ \ (n \in \mathbb{Z}), \\ [a_m, a_n] &= m\delta_{m,-n}\hbar \ \ (m, n \in \mathbb{Z}). \end{aligned} \tag{3.1}$$

Note that $[a_0, a_n] = 0$ for all $n \in \mathbb{Z}$ and so a_0 is a central element in $\mathcal{A}$. This algebra is often called the *oscillator* or *Heisenberg algebra*.

Much as in the last chapter, where we introduced the fermionic Fock space F for the purpose of constructing representations of the $g\ell_\infty, a_\infty$ and $\overline{a_\infty}$ algebras, we now introduce the *(bosonic) Fock space* $B = \mathbb{C}[x_1, x_2, \cdots; z, z^{-1}]$, the space of polynomials in infinitely many variables $x_1, x_2, \cdots$ along with z and z^{-1}.

We also define the charge m bosonic Fock space to be the subspace

$$B^{(m)} = \mathbb{C}[x_1, x_2, \cdots]z^m.$$

It is easy to see that each of the charge m bosonic Fock spaces are isomorphic.

We construct the representation $\hat{r}^B_m$ of $\mathcal{A}$ on $B^{(m)}$ for fixed m as follows. For all $n \geq 1$:

$$\begin{aligned} \hat{r}^B_m(a_n) &= \partial/\partial x_n, \\ \hat{r}^B_m(a_{-n}) &= n x_n, \\ \hat{r}^B_m(a_0) &= mI, \\ \hat{r}^B_m(\hbar) &= I, \end{aligned} \tag{3.2}$$

where I is the identity operator on $B^{(m)}$.

Lemma 3.1.1. *The representation* (3.2) *of $\mathcal{A}$ is irreducible.*

Proof. Any polynomial in $B^{(m)}$ can be reduced to a multiple of z^m by successive application of the a_n with $n > 0$ (these are sometimes called annihilation operators). Then successive application of the a_{-n} with $n > 0$ can give any other monomial in $B^{(m)}$ (this is why the a_{-n} with $n > 0$ are sometimes called creation operators). □

We call the constant polynomial $v = z^m$ the vacuum vector of $B^{(m)}$ and we see immediately that it enjoys the following properties:

$$\begin{aligned} a_n(v) &= 0 \qquad (n > 0), \\ a_0(v) &= mv, \\ \hbar(v) &= v. \end{aligned} \tag{3.3}$$

Proposition 3.1.2. *Let V be a representation of $\mathcal{A}$ which admits a non-zero vector v such that equations* (3.3) *are satisfied. Then the monomials $a_{-1}^{k_1} \cdots a_{-n}^{k_n}(v)$ with $k_i \in \mathbb{Z}_+$ are linearly independent. If these monomials span V then V is equivalent to the representation of $\mathcal{A}$ on $B^{(m)}$ given by* (3.2)*, for any fixed charge m. In particular, this is the case if V is irreducible.*

Proof. We have a mapping ϕ from $B^{(m)}$ to V defined by

$$\phi(P(\cdots, x_n, \cdots)z^m) = P(\cdots, \frac{a_{-n}}{n}, \cdots)v.$$

It is clear that if Pz^m is an element of $B^{(m)}$, then $a_n(\phi(Pz^m)) = \phi(a_n(Pz^m))$ so that ϕ is an intertwining operator. Since $B^{(m)}$ is irreducible, $\ker \phi = 0$ and so ϕ is an isomorphism if ϕ is onto. □

We may also define an anti-linear anti-involution on $\mathcal{A}$ by

$$\omega(a_n) = a_{-n}, \quad \omega(\hbar) = \hbar. \tag{3.4}$$

Proposition 3.1.3. *Let V be as in Proposition 3.1.2. Then V carries a unique Hermitian form $\langle\cdot|\cdot\rangle$ which is contravariant with respect to ω and such that $\langle v|v\rangle = 1$ for the vacuum vector v. The distinct monomials $a_{-1}^{k_1}\cdots a_{-n}^{k_n}v$ with $k_i \in \mathbb{Z}_+$ form an orthogonal basis with respect to the Hermitian form. These monomials have norms given by*

$$\langle a_{-1}^{k_1}\cdots a_{-n}^{k_n}v|a_{-1}^{k_1}\cdots a_{-n}^{k_n}v\rangle = \prod_{j=1}^{n} k_j!. \tag{3.5}$$

Proof. If $\langle\cdot|\cdot\rangle$ is a contravariant Hermitian form, then both the orthogonality and (3.5) are proved by induction on $k_1 + \cdots + k_n$, proving uniqueness. One checks directly that the Hermitian form, for which monomials are orthogonal and have norms given by (3.5), is contravariant, proving existence. □

Corollary 3.1.4. *The Hermitian form on V of Proposition 3.1.3 is positive definite.*

Definition 3.1.5. *Let Pz^m be an arbitrary element in $B^{(m)}$. The vacuum expectation value of P, denoted by (P), is defined as the constant term in P.*

We can see very quickly that

$$(\omega(P)) = \overline{(P)}.$$

Proposition 3.1.6. *For $P, Q \in B^{(m)}$ the Hermitian form*

$$\langle P|Q\rangle = (\omega(P)Q) = \overline{P}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{2\partial x_2}, \frac{\partial}{3\partial x_3}, \cdots\right) Q(x)\bigg|_{x_1=x_2=\cdots=0} \tag{3.6}$$

is equivalent to the Hermitian form given by (3.5).

Proof. It is easily checked that this is a Hermitian form which is contravariant and for which $\langle 1|1\rangle = 1$. Hence, by Proposition 3.1.3, formulas (3.6) and (3.5) are equivalent. □

3.2 The Boson - Fermion Correspondence (I)

Recall from the previous chapter that the shift operators Λ_k generate a commutative subalgebra of $\overline{a_\infty}$. The subalgebra of a_∞ generated by the shift operators and the central element c can be identified with the oscillator algebra $\mathcal{A}$. This can be seen by noticing that in a_∞,

$$[\Lambda_n, \Lambda_k] = \alpha(\Lambda_n, \Lambda_k)c.$$

It is then straightforward to compute

$$\alpha(\Lambda_n, \Lambda_k) = n\delta_{n,-k},$$

so that

$$[\Lambda_n, \Lambda_k] = n\delta_{n,-k}c.$$

Comparing this with (3.1), we see that this can indeed be identified with $\mathcal{A}$. Note that the anti-linear anti-involution ω of $\mathcal{A}$ is consistent with conjugate transpose on a_∞.

We will simultaneously view $\mathcal{A}$ as being the subalgebra of a_∞ generated by the shift operators and c, as well as the abstract algebra defined in (3.1). We will do this by making use of the identification $a_n \leftrightarrow \Lambda_n$ and $\hbar \leftrightarrow c$.

Since $\hat{r}_m$ is a unitary irreducible representation of a_∞, the restriction to the subalgebra generated by the shift operators and c gives us a representation of $\mathcal{A}$ on $F^{(m)}$. Explicitly, this representation is given by $\hat{r}_m(a_n) = \hat{r}_m(\Lambda_n)$ and $\hat{r}_m(\hbar) = \hat{r}_m(c)$.

Proposition 3.2.1. *The representation $\hat{r}_m$ of $\mathcal{A}$ on $F^{(m)}$ is irreducible and is isomorphic to the representation $\hat{r}^B_m$ of $\mathcal{A}$ on B given by* (3.2)*. This isomorphism is denoted σ_m.*

Proof. First, we see immediately that

$$\hat{r}_m(a_k)\psi_m = \hat{r}_m(\Lambda_k)\psi_m = 0 \text{ for } k > 0,$$
$$\hat{r}_m(a_0)\psi_m = \hat{r}_m(\Lambda_0)\psi_m = m\psi_m,$$

and that $\hat{r}_m(\hbar) = \hat{r}_m(c) = 1$.

If we consider all semi-infinite monomials in $F^{(m)}$ of the form

$$\hat{r}_m(\Lambda_{-k_s}) \cdots \hat{r}_m(\Lambda_{-k_1})\psi_m \quad (0 < k_1 \leq k_2 \leq \cdots \leq k_s) \tag{3.7}$$

then by Proposition 3.1.2 these vectors are all linearly independent.

Since $\hat{r}_m(E_{i,j})$ replaces v_j by v_i or gives 0 we see that it changes the energy of a semi-infinite monomial by $j - i$. Hence, each monomial of the form given in (3.7) with $\sum_i k_i = k$ lie in $F^{(m)}_k$ and they form a basis since there are exactly $p(k)$ of them, where $p(k)$ is the number of partitions of size k. The rest now follows from Proposition 3.1.2. □

Explicitly, Proposition 3.2.1 tells us that

$$\begin{aligned} \hat{r}^B_m(\Lambda_k) &= \frac{\partial}{\partial x_k}, \\ \hat{r}^B_m(\Lambda_{-k}) &= kx_k, \\ \hat{r}^B_m(\Lambda_0) &= m. \end{aligned} \tag{3.8}$$

To the energy decomposition of $F^{(m)}$ by subspaces $F^{(m)}_k$ of energy k, there corresponds the *principal gradation* of $B^{(m)}$:

$$B^{(m)} = \bigoplus_{k \in \mathbb{Z}_+} B^{(m)}_k,$$

defined by

$$\deg(x_j) = j.$$

Consider the contravariant Hermitian form on $B^{(m)}$ defined by

$$\langle P|Q\rangle = \langle \sigma_m^{-1} P|\sigma_m^{-1} Q\rangle$$

for $P, Q \in B^{(m)}$. It is easily seen that this Hermitian form has the properties

$$\langle 1|1\rangle = 1 \quad \text{and} \quad \hat{r}_m^B(\Lambda_k)^* = \hat{r}_m^B(\Lambda_{-k}) = \hat{r}_m^B(\Lambda_k^*),$$

following from (3.4) and the fact that σ_m is an isomorphism. Hence, by Proposition 3.1.6 and Proposition 3.1.3, we know that

$$\langle P|Q\rangle = \overline{P}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{2\partial x_2}, \frac{\partial}{3\partial x_3}, \cdots\right) Q(x)\bigg|_{x_1=x_2=\cdots=0}.$$

The remainder of this section will be concerned with determining the polynomials in $B^{(m)}$ that correspond to the semi-infinite monomials $v_{i_m} \wedge v_{i_{m-1}} \wedge \cdots$ of $F^{(m)}$ under σ_m.

First, let

$$\begin{aligned}\overline{GL}_\infty = \{A = (a_{i,j}) | i, j \in \mathbb{Z}, A \text{ is invertible and all but a finite number} \\ \text{of the } a_{i,j} - \delta_{i,j} \text{ with } i \geq j \text{ are } 0\}.\end{aligned}$$

Thus matrices in $\overline{GL}_\infty$ have only a finite number of nonzero elements above the principal diagonal. It is easily seen that matrix multiplication in $\overline{GL}_\infty$ is well defined since the sums involved are finite. The Lie algebra of $\overline{GL}_\infty$ is:

$$\overline{g\ell}_\infty = \{(a_{i,j}) | i, j \in \mathbb{Z}, \text{ all but a finite number of the } a_{i,j} \text{ with } i \geq j \text{ are } 0\}.$$

The group $\overline{GL}_\infty$ and its Lie algebra $\overline{g\ell}_\infty$ act not on V, but on a completion $\overline{V}$ of V defined as

$$\overline{V} = \left\{\sum_j c_j v_j | c_j = 0 \text{ for } j \gg 0\right\}.$$

It is easy to see that the representations R and r extend to representations of $\overline{GL}_\infty$ and $\overline{g\ell}_\infty$ on the same space F constructed from V. In particular, the formulas for r_m and R_m hold as before. The exponential map is defined on the whole of $\overline{g\ell}_\infty$ and we have

$$\exp r(a) = R(\exp a) \text{ for } a \in \overline{g\ell}_\infty. \tag{3.9}$$

Theorem 3.2.2. *For any $m \in \mathbb{Z}$,*

$$\sigma_m(v_{i_m} \wedge v_{i_{m-1}} \wedge \cdots) = S_{i_m-m, i_{m-1}-m+1, \cdots}(x),$$

where $i_m > i_{m-1} > \cdots$, $i_{m-k} = m-k$ for k sufficiently large and $S_{i_m-m,i_{m-1}-m+1,\cdots}(x)$ is the Schur polynomial corresponding to the partition $(i_m - m, i_{m-1} - m + 1, \cdots)$. This is the same partition obtained from the semi-infinite monomial via its Maya diagram.

Proof. We will prove this for $m = 0$ as the proof for arbitrary m is essentially the same.

Let

$$P(x) = \sigma_0(v_{i_0} \wedge v_{i_{-1}} \wedge \cdots).$$

We will compute

$$R_0^B(\exp(y_1\Lambda_1 + y_2\Lambda_2 + \cdots))P(x) = \sigma_0(R_0(\exp(y_1\Lambda_1 + y_2\Lambda_2 + \cdots))v_{i_0} \wedge v_{i_{-1}} \wedge \cdots) \tag{3.10}$$

and then compare the coefficient of the vacuum on the two sides of (3.10).

First we compute the left hand side of (3.10). Notice that in the bosonic picture, $r_0(\Lambda_k)$ is represented by $\partial/\partial x_k$ for $k > 0$ so that

$$R_0^B(\exp(y_1\Lambda_1 + y_2\Lambda_2 + \cdots)) = \exp \sum_{j \geq 1} y_j \frac{\partial}{\partial x_j}.$$

Now, let $F(y)$ denote the coefficient of 1 when this operator is applied to $P(x)$. Then

$$\begin{aligned} F(y) &= \exp\left(\sum_{j \geq 1} y_j \frac{\partial}{\partial x_j}\right) P(x)\Bigg|_{x=0} \\ &= P(x+y)\big|_{x=0} = P(y) \end{aligned}$$

and so the coefficient of the vacuum on the left hand side of (3.10) is $P(x)$.

We now compute the right hand side of (3.10). It is clear that if $a = y_1\Lambda_1 + y_2\Lambda_2 + \cdots$, then $a \in \overline{g\ell}_\infty$ and $\exp a \in \overline{GL}_\infty$. Hence, from the discussion preceding (3.9), it follows (see also equation (2.4)) that

$$R_m(A)(v_{i_m} \wedge v_{i_{m-1}} \wedge \cdots) = \sum_{j_m > j_{m-1} > \cdots} \left(\det A^{i_m, i_{m-1}, \cdots}_{j_m, j_{m-1}, \cdots}\right) v_{j_m} \wedge v_{j_{m-1}} \wedge \cdots, \tag{3.11}$$

where here $A^{i_m, i_{m-1}, \cdots}_{j_m, j_{m-1}, \cdots}$ denotes the sub-matrix of the infinite matrix A with rows indexed by $j_m, j_{m-1}, \cdots$ and columns indexed by $i_m, i_{m-1}, \cdots$. In particular, we can apply the above formula to $R(\exp a)$.

Now, using (1.18),

$$\begin{aligned} \exp\left(\sum_{k \geq 1} \Lambda_k y_k\right) &= \exp\left(\sum_{k \geq 1} \Lambda_1^k y_k\right) \\ &= \sum_{k \geq 0} \Lambda_1^k S_k(y) \qquad (3.12) \\ &= \sum_{k \geq 0} \Lambda_k S_k(y) \end{aligned}$$

where the $S_k(y)$ are the elementary Schur polynomials. The right hand side of (3.12) can be viewed as a matrix A with matrix elements

$$A_{m,n} = S_{n-m}(y), \quad (m, n \in \mathbb{Z}).$$

Since $S_k(y) = 0$ for $k < 0$ we see that $A \in \overline{GL}_\infty$. Hence the right hand side of (3.10) becomes

$$\sigma_0(R(A)(v_{i_0} \wedge v_{i_{-1}} \wedge \cdots)).$$

Using (3.11) we then see that the required coefficient is

$$\det\left(A^{i_0, i_{-1}, i_{-2}, \cdots}_{0,-1,-2,\cdots}\right).$$

From (1.17) we see that this is simply $S_{i_0, i_{-1}+1, \cdots}(x)$ and so

$$P(x) = S_{i_0, i_{-1}+1, \cdots}(x),$$

as required. □

Theorem 3.2.2 tells us that each semi-infinite monomial $\psi \in F^{(m)}$ is mapped, under σ_m, to the Schur polynomial indexed by the partition corresponding to the monomial, that is, the partition formed by comparing the Maya diagram of ψ with the Maya diagram of ψ_m.

Corollary 3.2.3.

$$R^B_m(A) S_\lambda = \sum_\mu \det\left(A^{\lambda_1+m, \lambda_2+m-1, \cdots}_{\mu_1+m, \mu_2+m-1, \cdots}\right) S_\mu$$

where the sum is over partitions μ.

Proof. This result is an immediate consequence of Theorem 3.2.2 applied to (3.11). □

Corollary 3.2.4. *The Schur polynomials form an orthonormal basis in B with respect to the contravariant Hermitian form $\langle\cdot|\cdot\rangle$, i.e.*

$$\langle S_\lambda | S_\mu \rangle = \delta_{\lambda,\mu}.$$

Proof. This follows since σ_0 is an isomorphism and the corresponding semi-infinite monomials are orthonormal. □

Corollary 3.2.4 tells us that the Hermitian form on B coincides with the standard inner product on Schur polynomials.

3.3 The Boson-Fermion Correspondence (II)

In the previous section we looked at the isomorphism

$$\sigma_m : F^{(m)} \to B^{(m)},$$

which gives us a representation of the subalgebra $\mathcal{A}$ of a_∞ on $B^{(m)}$. We now extend this representation to the whole Lie algebra a_∞.

First, we prefer to work with F rather than each $F^{(m)}$ and so we define the direct sum of maps

$$\sigma = \bigoplus_{m\in\mathbb{Z}} \sigma_m,$$

so that

$$\sigma : F = \bigoplus_{m\in\mathbb{Z}} F^{(m)} \to B = \bigoplus_{m\in\mathbb{Z}} B^{(m)}. \tag{3.13}$$

Similar to the definition of $\hat{r}_m^B$, we define $\hat{r}^B(a) = \sigma\hat{r}(a)\sigma^{-1}$ for $a \in a_\infty$.

We now turn to the problem of describing explicitly the action of $\hat{v}_j$ and $\check{v}_j^*$ on B. First we introduce the generating functions

$$X(u) = \sum_{j\in\mathbb{Z}} u^j \hat{v}_j, \quad X^*(u) = \sum_{j\in\mathbb{Z}} u^{-j} \check{v}_j^*,$$

where u is an indeterminate which can be thought of as a nonzero complex number. It turns out that computing the transformation of both X and X^* under σ is easier to do than computing each of the wedging and contracting operators individually.

Something to note, however, is that $X(u)$ as an operator acting on $F^{(m)}$ does not map into $F^{(m+1)}$ but rather into the formal completion $\hat{F}^{(m+1)}$ where formal infinite sums of semi-infinite monomials are allowed Similarly, $X^*(u)$ maps $F^{(m)}$ into $\hat{F}^{(m-1)}$. We define

$$\hat{F} = \bigoplus_{m\in\mathbb{Z}} \hat{F}^{(m)}.$$

The transported operators $\sigma X(u)\sigma^{-1}$ and $\sigma X^*(u)\sigma^{-1}$ map B into $\hat{B}$, where $\hat{B}$ is the space of formal power series in $x_1, x_2, \cdots$ with coefficients which are polynomial in z, z^{-1}.

Recall from the previous chapter that

$$\begin{aligned}[\hat{r}(\Lambda_j), \hat{v}_k] &= \hat{v}_{k-j},\\ [\hat{r}(\Lambda_j), \check{v}_k^*] &= -\check{v}_{k+j}^*,\end{aligned}$$

from which we obtain

$$\begin{aligned}[\hat{r}(\Lambda_j), X(u)] &= u^j X(u),\\ [\hat{r}(\Lambda_j), X^*(u)] &= -u^j X^*(u).\end{aligned} \tag{3.14}$$

These equations hold in $\hat{F}$ and under the isomorphism $\sigma : \hat{F} \tilde{\to} \hat{B}$ they hold in $\hat{B}$ as well. We already know the transform of Λ_j. In particular, if $j > 0$, then

$$\begin{aligned} \hat{r}^B(\Lambda_j) &= \sigma \hat{r}(\Lambda_j) \sigma^{-1} = \frac{\partial}{\partial x_j}, \\ \hat{r}^B(\Lambda_{-j}) &= \sigma \hat{r}(\Lambda_{-j}) \sigma^{-1} = j x_j. \end{aligned} \tag{3.15}$$

We now define the operators $\Gamma(u)$ and $\Gamma^*(u)$ by

$$\begin{aligned} \Gamma(u) &= \sigma X(u) \sigma^{-1}, \\ \Gamma^*(u) &= \sigma X^*(u) \sigma^{-1}, \end{aligned} \tag{3.16}$$

so that our goal is now to determine an explicit description of the operators $\Gamma(u)$ and $\Gamma^*(u)$. The operators $\Gamma(u), \Gamma^*(u), X(u)$ and $X^*(u)$ are sometimes called vertex operators for reasons which will not be discussed here.

Using (3.14), (3.15) and (3.16) we see that

$$\begin{aligned} [\partial / \partial x_j, \Gamma(u)] &= u^j \Gamma(u), \\ [x_j, \Gamma(u)] &= \frac{u^{-j}}{j} \Gamma(u), \end{aligned}$$

along with the related relations for $\Gamma^*(u)$. These commutation relations are enough to determine $\Gamma(u)$ and $\Gamma^*(u)$ as is shown in the following proposition.

Proposition 3.3.1. *The operators $\Gamma(u)$ and $\Gamma^*(u)$ have the following form on $\hat{B}^{(m)}$:*

$$\begin{aligned} \Gamma(u)|_{\hat{B}^{(m)}} &= u^{m+1} z \exp\left(\sum_{j \geq 1} u^j x_j \right) \exp\left(- \sum_{j \geq 1} \frac{u^{-j}}{j} \frac{\partial}{\partial x_j} \right), \\ \Gamma^*(u)|_{\hat{B}^{(m)}} &= u^{-m} z^{-1} \exp\left(- \sum_{j \geq 1} u^j x_j \right) \exp\left(\sum_{j \geq 1} \frac{u^{-j}}{j} \frac{\partial}{\partial x_j} \right). \end{aligned}$$

Proof. We only prove the result for $\Gamma(u)$ since the proof for $\Gamma^*(u)$ is almost exactly the same.

First notice that the z factor needs to be present on the right hand side since $\Gamma(u)$ maps $\hat{B}^{(m)}$ into $\hat{B}^{(m+1)}$. Let T_u be the operator on $\hat{B}$ defined by

$$T_u = \exp\left(\sum_{j \geq 1} \frac{u^{-j}}{j} \frac{\partial}{\partial x_j} \right).$$

Using Taylor's formula it can be seen that for any $f \in \hat{B}$

$$(T_u f)(x_1, x_2, \cdots) = f\left(x_1 + u^{-1}, x_2 + \frac{u^{-2}}{2}, \cdots, x_j + \frac{u^{-j}}{j}, \cdots \right).$$

Since

$$T_u(x_j f) = \left(x_j + \frac{u^{-j}}{j}\right) T_u f,$$

we see that

$$[x_j, T_u] = \frac{-u^{-j}}{j} T_u.$$

Using this and the relation

$$[x_j, \Gamma(u)] = \frac{u^{-j}}{j}\Gamma(u),$$

we see that

$$\begin{aligned} x_j \Gamma(u) T_u &= \Gamma(u) x_j T_u + \frac{u^{-j}}{j}\Gamma(u) T_u \\ &= \Gamma(u) T_u x_j + \frac{u^{-j}}{j}\Gamma(u) T_u - \frac{u^{-j}}{j}\Gamma(u) T_u \\ &= \Gamma(u) T_u x_j, \end{aligned}$$

and so

$$[x_j, \Gamma(u) T_u] = 0.$$

From this we can conclude that $\Gamma(u)T_u$ contains no differential operators and so is a power series. This tells us that

$$\Gamma(u) = z f(x_1, x_2, \cdots) \exp\left(-\sum_{j\geq 1} \frac{u^{-j}}{j}\frac{\partial}{\partial x_j}\right),$$

where $f(x_1, x_2, \cdots)$ still needs to be determined. Using the relations

$$\left[\frac{\partial}{\partial x_j}, \Gamma(u)\right] = u^j \Gamma(u),$$

$$\left[\frac{\partial}{\partial x_j}, \exp\left(-\sum_{j\geq 1} u^j x_j\right)\right] = -u^j \exp\left(-\sum_{j\geq 1} u^j x_j\right),$$

we see that

$$\left[\frac{\partial}{\partial x_j}, \exp\left(-\sum_{j\geq 1} u^j x_j\right)\Gamma(u)\right] = 0,$$

from which we conclude that

$$\Gamma(u) = c_m(u) z \exp\left(\sum_{j\geq 1} u^j x_j\right) \exp\left(-\sum_{j\geq 1} \frac{u^{-j}}{j}\frac{\partial}{\partial x_j}\right).$$

Lastly, to determine $c_m(u)$, we note that the coefficient of the vacuum vector ψ_{m+1} of $\hat{F}^{(m+1)}$ in the expansion of $X(u)\psi_m$ is u^{m+1}. This completes the proof. $\square$

Definition 3.3.2. *The operator $R(u) : \hat{B} \to \hat{B}$ is defined by*

$$R(u)f(x;z) = uzf(x;uz).$$

We see that if $f(x;z) = z^m g(x_1, x_2, \cdots)$ then

$$R(u)f(x;z) = u^{m+1}z^{m+1}g(x_1, x_2, \cdots).$$

Using this we can write down the general form of $\Gamma(u)$ and $\Gamma^*(u)$.

Theorem 3.3.3. *We can write $\Gamma(u)$ and $\Gamma^*(u)$ as*

$$\Gamma(u) = R(u)\exp\left(\sum_{j\geq 1} u^j x_j\right)\exp\left(-\sum_{j\geq 1}\frac{u^{-j}}{j}\frac{\partial}{\partial x_j}\right),$$
$$\Gamma^*(u) = R(u)^{-1}\exp\left(-\sum_{j\geq 1} u^j x_j\right)\exp\left(\sum_{j\geq 1}\frac{u^{-j}}{j}\frac{\partial}{\partial x_j}\right).$$

In the literature Theorem 3.2.2 and Theorem 3.3.3 together are referred to as the Boson-Fermion correspondence.

3.4 Realizing the $GL_\infty\psi_0$ orbit

Recall that in Theorem 2.6.1 we characterized elements in F which are in the $GL_\infty\psi_0$ orbit. Now we would like to make use of Theorem 2.6.1 to give a characterization of the elements in $B^{(0)}$ which correspond to elements in $GL_\infty\psi_0$. We do this by making use of the Boson-Fermion correspondence developed in the first portion of this chapter. The functions in $B^{(0)}$ which correspond to elements in $GL_\infty\psi_0$ are called *tau functions* of the KP hierarchy.

Consider the expression

$$X(u)\tau \otimes X^*(u)\tau. \tag{3.17}$$

Upon expanding (3.17) we get

$$\sum_{i,j} u^{i-j}\hat{v}_i(\tau)\otimes\check{v}_j^*(\tau),$$

and it follows from Theorem 2.6.1 that $\tau \in \sigma_0(GL_\infty\psi_0)$ if and only if the term independent of u in (3.17) vanishes.

The isomorphism

$$\sigma_0 : F^{(0)} \tilde{\to} \mathbb{C}[x_1, x_2, \cdots]$$

extends to an isomorphism between $F^{(0)} \otimes F^{(0)}$ and $\mathbb{C}[x_1', x_2', \cdots; x_1'', x_2'', \cdots]$, which is the polynomial ring in two infinite sets of variables, $x_1', x_2', \cdots$, and $x_1'', x_2'', \cdots$. We can transform (3.17) to the bosonic representation using the identification

$$X(u) \mapsto \Gamma(u) = uz \exp\left(\sum_{j\geq 1} u^j x_j'\right) \exp\left(-\sum_{j\geq 1} \frac{u^{-j}}{j} \frac{\partial}{\partial x_j'}\right),$$
$$X^*(u) \mapsto \Gamma^*(u) = z^{-1} \exp\left(-\sum_{j\geq 1} u^j x_j''\right) \exp\left(\sum_{j\geq 1} \frac{u^{-j}}{j} \frac{\partial}{\partial x_j''}\right).$$

So, (3.17) becomes

$$u \exp\left(\sum_{j\geq 1} u^j (x_j' - x_j'')\right) \exp\left(-\sum_{j\geq 1} \frac{u^{-j}}{j} \left(\frac{\partial}{\partial x_j'} - \frac{\partial}{\partial x_j''}\right)\right) \tau(x')\tau(x''). \qquad (3.18)$$

Defining new variables x_i, y_i by

$$x_i' = x_i - y_i, \quad x_i'' = x_i + y_i,$$

so that

$$x_i' - x_i'' = -2y_i, \quad \frac{\partial}{\partial x_i'} - \frac{\partial}{\partial x_i''} = -\frac{\partial}{\partial y_i},$$

we deduce the following result from Theorem 2.6.1.

Proposition 3.4.1. *A nonzero element τ of $\mathbb{C}[x_1, x_2, \cdots]$ is contained in $\sigma_0\left(GL_\infty \psi_0\right)$ if and only if the coefficient of u^0 vanishes in the expression*

$$u \exp\left(-\sum_{j\geq 1} 2u^j y_j\right) \exp\left(\sum_{j\geq 1} \frac{u^{-j}}{j} \frac{\partial}{\partial y_j}\right) \tau(x-y)\tau(x+y). \qquad (3.19)$$

It can be seen that the above formula gives rise to a countable number of partial differential equations which must be satisfied by τ. In the next section we will look at this in detail and write out some of the equations that arise in this way.

3.5 Hirota's bilinear equations

Definition 3.5.1. *Given a polynomial $P(x_1, x_2, \cdots)$ and two functions f and g, we denote by $Pf \cdot g$ the expression*

$$P\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \cdots\right) \left(f(x_1 - u_1, x_2 - u_2, \cdots) g(x_1 + u_1, x_2 + u_2, \cdots)\right)\Big|_{u=0}.$$

The equation $Pf \cdot g = 0$ is called a *Hirota bilinear equation.*

Example 3.5.2. *Let $P = x_1$. Then*

$$Pf \cdot g = \frac{\partial}{\partial u_1}(f(x_1 - u_1)g(x_1 + u_1))\bigg|_{u_1=0} = -g(x_1)\frac{\partial f}{\partial x_1} + f(x_1)\frac{\partial g}{\partial x_1}.$$

Example 3.5.3. *Let $P = x_1^n$. Then from Leibniz's formula we get*

$$Pf \cdot g = \sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{\partial^k f}{\partial x_1^k} \frac{\partial^{n-k} g}{\partial x_1^{n-k}}.$$

Note that $Pf \cdot f = 0$ if $P(x_1, x_2, \cdots) = -P(-x_1, -x_2, \cdots)$. In other words, $Pf \cdot f = 0$ if P is odd. Since the Hirota bilinear equations that arise in the KP hierarchy are of the form $Pf \cdot f = 0$ and are linear in P, this implies that we need only consider the even terms in P.

Expanding (3.19) using the generating function for elementary Schur polynomials we get

$$u \left(\sum_{j \geq 0} u^j S_j(-2y) \right) \left(\sum_{j \geq 0} u^{-j} S_j(\tilde{\partial}_y) \right) \tau(x - y)\tau(x + y),$$

where

$$\tilde{\partial}_y = \left(\frac{\partial}{\partial y_1}, \frac{\partial}{2\partial y_2}, \frac{\partial}{3\partial y_3}, \cdots \right).$$

Setting the term independent of u equal to zero, we get the following system of equations:

$$\sum_{j \geq 0} S_j(-2y) S_{j+1}(\tilde{\partial}_y) \tau(x - y)\tau(x + y) = 0.$$

Notice that

$$\begin{aligned} S_{j+1}(\tilde{\partial}_y)\tau(x - y)\tau(x + y) &= S_{j+1}(\tilde{\partial}_u)\tau(x - y - u)\tau(x + y + u)\Big|_{u=0} \\ &= S_{j+1}(\tilde{\partial}_u) \exp\left(\sum_{s \geq 1} y_s \frac{\partial}{\partial u_s} \right) \tau(x - u)\tau(x + u)\Bigg|_{u=0}, \end{aligned}$$

using Taylor's formula. However, the last part can be written as a family of Hirota bilinear equations (each equation corresponding to a coefficient of some monomial in the y_i's):

$$S_{j+1}(\tilde{x}) \exp\left(\sum_{s \geq 1} y_s x_s \right) \tau(x) \cdot \tau(x),$$

where

$$\tilde{x} = \left(x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \cdots \right).$$

Thus we get:

Theorem 3.5.4. *A nonzero polynomial τ is contained in $\sigma_0\left(GL_\infty\psi_0\right)$ if and only if τ is a solution of the following system of Hirota bilinear equations:*

$$\sum_{j=0}^{\infty} S_j(-2y)S_{j+1}(\tilde{x})\exp\left(\sum_{s\geq 1} y_s x_s\right)\tau(x)\cdot\tau(x) = 0, \tag{3.20}$$

where $y_1, y_2, \cdots$ are free parameters.

We can now use Theorem 3.5.4 to construct a countable number of partial differential equations which have the property that a polynomial satisfies them all simultaneously if and only if the polynomial is contained in $\sigma_0\left(GL_\infty\psi_0\right)$. Each of these partial differential equations is obtained by extracting the coefficient of some monomial in the y_i's. As an example, we can construct the partial differential equations related to the monomials y_r for some small values of r.

Expanding the exponential in (3.20) in Theorem 3.5.4 we see that y_r appears exactly once with coefficient x_r. In the expansion of the $S_j(-2y)$, y_r appears only in $S_r(-2y)$ with coefficient -2. Thus we get the Hirota bilinear equation

$$(x_r x_1 - 2S_{r+1}(\tilde{x}))\tau\cdot\tau = 0.$$

Using the expansion of S_r for some small values of r (see Example 1.3.3) we see that

$$\begin{aligned}
x_1x_1 - 2S_2(\tilde{x}) &= -x_2,\\
x_2x_1 - 2S_3(\tilde{x}) &= -\frac{x_1^3}{3} - \frac{2x_3}{3},\\
x_3x_1 - 2S_4(\tilde{x}) &= \frac{x_1x_3}{3} - \frac{x_4}{2} - \frac{x_2^2}{4} - \frac{x_1^4}{12} - \frac{x_1^2x_2}{2}.
\end{aligned}$$

Since we know that odd polynomials give us trivial Hirota equations, we see that for $r = 1, 2$ the above equations are trivial and for $r = 3$ the even terms give us the Hirota equation

$$(x_1^4 + 3x_2^2 - 4x_1x_3)\tau\cdot\tau = 0.$$

This then becomes

$$\left(\frac{\partial^4}{\partial u_1^4} + 3\frac{\partial^2}{\partial u_2^2} - 4\frac{\partial^2}{\partial u_1\partial u_3}\right)\tau(x+u)\tau(x-u)\Big|_{u=0} = 0. \tag{3.21}$$

If we then put $x_1 = x, x_2 = y, x_3 = t$ and introduce a new function

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}(\log\tau),$$

then equation (3.21) becomes

$$\frac{3}{4}\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} - \frac{3}{2}u\frac{\partial u}{\partial x} - \frac{1}{4}\frac{\partial^3 u}{\partial x^3}\right). \tag{3.22}$$

Equation (3.22) is classically known as the *Kadomtzev - Petviashvili (KP) equation*. Since this is the simplest equation in the hierarchy of partial differential equations given in Theorem 3.5.4 the entire hierarchy is called the *Kadomtzev - Petviashvili (or KP) hierarchy*.

Chapter 4

A Symmetric Function Approach

In this chapter we take an alternate approach to the bilinear form (3.18) of the KP hierarchy by reinterpreting the equations in the context of symmetric functions. For the most part we will be presenting a new combinatorial proof of a classical result for the KP hierarchy which is described in [3].

4.1 Codes of partitions

Before we begin looking at the KP hierarchy in terms of symmetric functions we need to describe an alternate way of encoding partitions which is based on the Maya diagrams for semi-infinite monomials described in chapter two.

We can represent any semi-infinite monomial ψ as a two-colouring (white and black) of the integers with the property that for some positive integer N all integers $k > N$ are white and all integers $k < -N$ are black. When compared to a unique vacuum vector ψ_m, (where m is the charge of ψ) we can represent this colouring uniquely with a partition. In other words, semi-infinite monomials are completely specified by a partition and a charge.

Since solutions of the KP hierarchy are elements in $GL_\infty\psi_0$, we need only consider elements with charge 0, and every semi-infinite monomial of charge 0 is uniquely specified by a partition. In terms of Maya diagrams, fixing the charge corresponds to considering two-colourings of the integers up to arbitrary shifts.

Rather than viewing a partition in terms of its Maya diagram, we view it as a two-way infinite binary string in two symbols, say U and R, where the string is infinitely U to the left and infinitely R to the right. In this case we can view the string as a set of instructions detailing how to reconstruct the partition. The symbol U denotes moving one unit upwards and R denotes moving one unit to the right. The resulting path then marks the outline of the Young diagram of the partition in question. Note that this is different from the Maya diagram in that we only view the two-colouring up to arbitrary shifts.

If λ is a partition and σ is the binary string corresponding to λ then we say that σ is the *code* of λ. (see, e.g., [31, p. 467] for more on codes; in [31], the symbols in the code are 0, 1, but we prefer U, R since they are more mnemonic.)

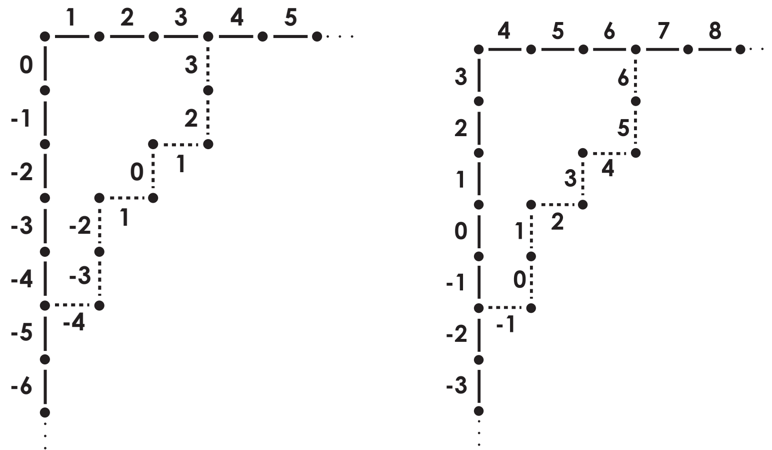

Figure 4.1: Semi-Infinite Monomials With Different Charges

Example 4.1.1. *Consider the semi-infinite monomial* $\psi = v_3 \wedge v_2 \wedge v_0 \wedge v_{-2} \wedge v_{-3} \wedge v_{-5} \wedge \cdots$. *When we compare it to* ψ_0 *we arrive at the diagram on the left in Figure 4.1. Likewise, the semi-infinite monomial* $\gamma = v_6 \wedge v_5 \wedge v_3 \wedge v_1 \wedge v_0 \wedge v_{-2} \wedge \cdots$ *when compared to* ψ_3 *gives rise to the diagram on the right in Figure 4.1. Both of these diagrams outline the same partition,* $3^2 21^2$*, the only difference being the charge (the label on the up step in the upper left corner of the diagram). Since we only want to record the shape of the partition we remove the integer labeling and simply record the string of up and down steps that outline the partition. In the example of* $3^2 21^2$ *above we are left with the two-way infinite binary string* $\cdots UUURUURURUURRR \cdots$ *which is shown in Figure 4.2.*

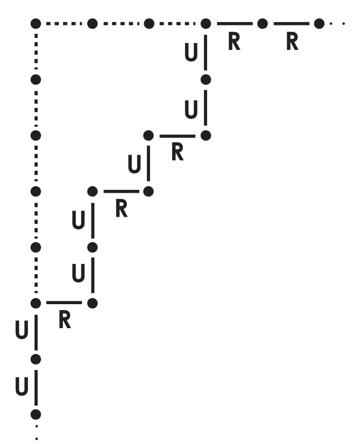

Figure 4.2: The Code Of $3^2 21^2$

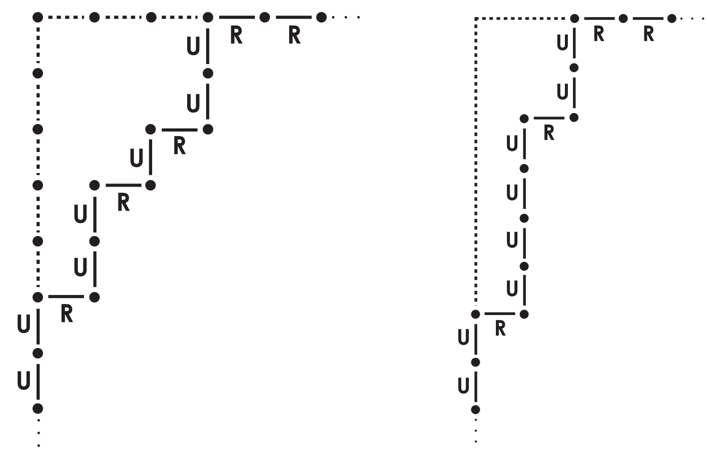

Figure 4.3: $3^2 21^2$ and $(3^2 21^2)^{(2)}$

We let $\lambda^{(i)}$, $i \geq 1$, be the partition whose code is obtained from the code of the partition λ by switching the ith R (from the left) to U. If $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$, then we immediately have

$$\lambda^{(i)} = (\lambda_1 - 1, \cdots \lambda_j - 1, i - 1, \lambda_{j+1}, \cdots, \lambda_n), \tag{4.1}$$

where j is chosen (uniquely) from $0, \cdots, n$ so that $\lambda_j \geq i > \lambda_{j+1}$ (with the conventions that $\lambda_{n+1} = 0$ and $\lambda_0 = \infty$). Now, define $u_i(\lambda)$ to be the number of up-steps U that follow the ith right-step R from the left in the code of λ. Then note that $u_i(\lambda) = j$, and that we have $|\lambda^{(i)}| = |\lambda| - j + i - 1$ from (4.1), so we can determine $u_i(\lambda) = j$ in terms of i via

$$u_i(\lambda) = |\lambda| - |\lambda^{(i)}| + i - 1. \tag{4.2}$$

Also note that $u_i(\lambda)$ weakly decreases as i increases, so we also obtain

$$|\lambda| - l(\lambda) = |\lambda^{(1)}| < |\lambda^{(2)}| < \cdots, \qquad |\lambda^{(i)}| = |\lambda| + i - 1, i > \lambda_1. \tag{4.3}$$

Example 4.1.2. *Consider the partition* $3^2 21^2$ *whose code is* $\cdots UURUURURUURR \cdots$ *(pictured on the left in Figure 4.3). Computing the code corresponding to* $(3^2 21^2)^{(2)}$ *we get* $\cdots UURUUUURUURR \cdots$ *which is pictured on the right in Figure 4.3.*

Alternatively, from equation (4.1)*, we see that* $(3^2 21^2)^{(2)}$ *is constructed by adding a part of size* 1 *and decreasing all parts larger than the new part. This tells us that* $(3^2 21^2)^{(2)} = 2^2 1^4$ *as we computed earlier and which is shown in Figure 4.3.*

We also let $\lambda^{(-i)}$, $i \geq 1$, be the partition whose code is obtained from the code of the partition λ by switching the ith U (from the right) to R. If $\lambda = (\lambda_1, \lambda_2, \cdots)$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ (i.e., λ has finitely many positive parts and an infinite number of trailing 0's), then we immediately have

$$\lambda^{(-i)} = (\lambda_1 + 1, \cdots, \lambda_{i-1} + 1, \lambda_{i+1}, \cdots). \tag{4.4}$$

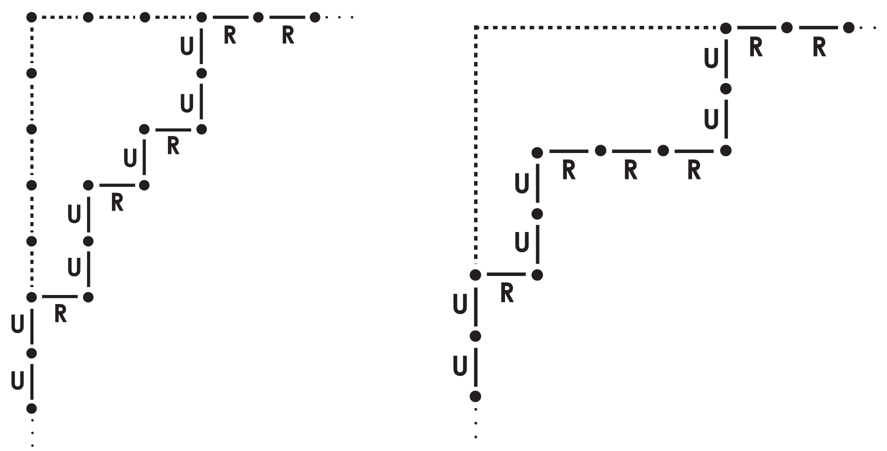

Figure 4.4: $3^2 21^2$ and $(3^2 21^2)^{(-3)}$

Thus we have $|\lambda^{(-i)}| = |\lambda| - \lambda_i + i - 1$. Since λ_i weakly decreases as i increases, we obtain

$$|\lambda| - \lambda_1 = |\lambda^{(-1)}| < |\lambda^{(-2)}| < \cdots , \qquad |\lambda^{(-i)}| = |\lambda| + i - 1, \quad i > l(\lambda). \tag{4.5}$$

Example 4.1.3. *As before we begin with the partition* $3^2 21^2$ *pictured on the left in Figure 4.4 and which has the code* $\cdots UURUURURUURR \cdots$. *We compute* $(3^2 21^2)^{(-3)}$ *which has the code* $\cdots UURUURRRUURR \cdots$, *and is pictured on the right in Figure 4.4.*

Alternatively, from equation (4.4)*, we see that* $(3^2 21^2)^{(-3)}$ *corresponds to removing the part of size* 2 *and increasing the size of each part above it. In other words,* $(3^2 21^2)^{(-3)} = 4^2 2^2$*, which is shown in Figure 4.4.*

Recall that given a partition λ, its conjugate λ' is formed by flipping its diagram along the main diagonal. In terms of the code of λ this corresponds to reversing the order of the code and swapping the U's and R's. It is easy to see that

$$\omega(\lambda^{(i)}) = \omega(\lambda)^{(-i)}$$

where ω is the fundamental involution (with action $\omega(\lambda) = \lambda'$).

4.2 The Bernstein Operator

First, recall from (3.18) that the bilinear form of the KP hierarchy tells us that a polynomial τ is a solution to the KP hierarchy (or is a tau function of the KP hierarchy) if and only if

$$[u^{-1}]\exp\left(\sum_{j\geq 1} u^j x'_j\right)\exp\left(-\sum_{j\geq 1}\frac{u^{-j}}{j}\frac{\partial}{\partial x'_j}\right)\tau(x')$$
$$\exp\left(-\sum_{j\geq 1} u^j x''_j\right)\exp\left(\sum_{j\geq 1}\frac{u^{-j}}{j}\frac{\partial}{\partial x''_j}\right)\tau(x'')=0.$$

Equivalently, let

$$\Phi(x';u)=\exp\left(\sum_{j\geq 1} u^j x'_j\right)\exp\left(-\sum_{j\geq 1}\frac{u^{-j}}{j}\frac{\partial}{\partial x'_j}\right),$$
$$\Phi^{\perp}(x'';u)=\exp\left(-\sum_{j\geq 1} u^j x''_j\right)\exp\left(\sum_{j\geq 1}\frac{u^{-j}}{j}\frac{\partial}{\partial x''_j}\right).$$

Then τ is a solution to the KP hierarchy if and only if

$$[u^{-1}](\Phi(x',u)\tau(x'))(\Phi^{\perp}(x'',u)\tau(x''))=0.$$

Recall that the power sum symmetric functions p_k form a set of algebraically independent generators for the ring of symmetric functions Λ. We can make the change of variables

$$x'_j \mapsto \frac{p_j}{j}, \qquad \frac{\partial}{\partial x'_j}\mapsto j\frac{\partial}{\partial p_j}. \tag{4.6}$$

In what follows we always write symmetric functions without specifying the variables in which they are written. Under this convention we assume that the symmetric functions written unadorned (i.e., e_k, h_k, s_λ, p_k, etc.) are in terms of the indeterminates x'_i and that marked symmetric functions ($\hat{e}_k, \hat{h}_k, \hat{s}_\lambda, \hat{p}_k$, etc.) are written in terms of the indeterminates x''_i.

Under the change of variables (4.6), the operators Φ and $\Phi^{\perp}$ become

$$\Phi(x';u)\mapsto B(p;u)=\exp\left(\sum_{j\geq 1}\frac{u^j}{j}p_j\right)\exp\left(-\sum_{j\geq 1}u^{-j}\frac{\partial}{\partial p_j}\right),$$
$$\Phi^{\perp}(x'';u)\mapsto B^{\perp}(\hat{p};u)=\exp\left(-\sum_{j\geq 1}\frac{u^j}{j}\hat{p}_j\right)\exp\left(\sum_{j\geq 1}u^{-j}\frac{\partial}{\partial \hat{p}_j}\right).$$

The operator $B(p;u)$ was originally introduced by Bernstein [33, p. 69] in the study of modular representations of the symmetric group. See also Macdonald [22, p. 95]. We call $B(p;u)$ the *Bernstein operator* and $B^{\perp}(\hat{p};u)$ the *adjoint Bernstein operator* for reasons given below.

Recall that

$$\sum_{i\geq 0} h_i t^i=\exp\sum_{k\geq 1}\frac{p_k}{k}t^k, \qquad \sum_{i\geq 0} e_i t^i=\exp\sum_{k\geq 1}\frac{p_k}{k}(-1)^{k-1}t^k, \tag{4.7}$$

where the h_i are the complete symmetric functions and the e_i are the elementary symmetric functions. Also recall that

$$p_k^\perp = k\frac{\partial}{\partial p_k}, \tag{4.8}$$

where $p_k^\perp$ is the adjoint of multiplication by p_k with respect to the standard inner product. Using (4.7) and (4.8) we can rewrite the operators B and $B^\perp$ as

$$B(p;u) = \sum_{n\in\mathbb{Z}} B_n u^n = \sum_{k,m\geq 0} (-1)^m u^{k-m} h_k e_m^\perp, \tag{4.9}$$

and,

$$B^\perp(\hat{p};u) = \sum_{n\in\mathbb{Z}} B_n^\perp u^n = \sum_{k,m\geq 0} (-1)^m u^{m-k} \hat{e}_m \hat{h}_k^\perp. \tag{4.10}$$

This gives us immediately that $B_n^\perp = (-1)^n \omega B_{-n}\omega$ and so $B^\perp(p;u) = \omega B(p;-u)\omega$ where ω acts on a series in u by acting on its coefficients (which are symmetric functions). Thus, for much of what follows we only consider $B(p;u)$ since analogous statements may be made for $B^\perp(p;u)$.

Our goal will be to understand the action of the operator B on Schur functions and to this end we first look at the following set of combinatorial objects. For partitions λ and μ, let $\mathcal{R}_{\lambda,\mu}$ be the set of partitions ν such that $\lambda - \nu$ is a vertical strip, and $\mu - \nu$ is a horizontal strip. Also, let

$$R_{\lambda,\mu} = \sum_{\nu\in\mathcal{R}_{\lambda,\mu}} (-1)^{|\lambda|-|\nu|}, \tag{4.11}$$

which is 0 when $\mathcal{R}_{\lambda,\mu}$ is empty.

Proposition 4.2.1. *For any partition λ, we have*

$$B(p;u)s_\lambda = \sum_{\mu\in\mathcal{P}} R_{\lambda,\mu} u^{|\mu|-|\lambda|} s_\mu,$$

where $\mathcal{P}$ is the set of partitions.

Proof. Recall that

$$h_n s_\lambda = \sum_\mu s_\mu \tag{4.12}$$

where the sum is over partitions μ such that $\mu - \lambda$ is a horizontal n-strip and

$$e_n^\perp s_\lambda = \sum_\mu s_\mu \tag{4.13}$$

where the sum is over partitions μ such that $\lambda - \mu$ is a vertical n-strip.

Putting formulas (4.12) and (4.13) together with the formula for the $B(p;u)$ operator (4.9) immediately gives us the result. □

We can view the operator $B(p;u)$ as acting in two steps. The first is to apply the operator $e_m^\perp$ and the second is to multiply by h_k. Since we wish to consider the action of $B(p;u)$ on the Schur function s_λ this corresponds to first removing a vertical strip (of size m) and adding a horizontal strip (of size k) to the result. The coefficient for some s_μ in the resulting series will contain u^{k-m}, which has its exponent equal to the net difference in size from λ to μ, and the sum over all possible ways of getting to μ from λ in the way described weighted by $(-1)^m$. From the definition of $\mathcal{R}_{\lambda,\mu}$ we see that $R_{\lambda,\mu}$ is the appropriately weighted sum of partitions.

There are three classes of squares in the union of the diagrams for λ and μ that we shall consider when $\mathcal{R}_{\lambda,\mu}$ is nonempty:

- The squares of μ that are not contained in λ. These squares are necessarily bottommost in their column of μ. None of these is contained in any ν in $\mathcal{R}_{\lambda,\mu}$. Such squares are contained in the horizontal strip that is added in the multiplication by an h_k. In other words, these are squares which are added by $B(p;u)$ but which could not have first been removed.

- The squares of λ that are not contained in μ. These squares are necessarily rightmost in their row of λ. None of these squares is contained in any ν in $\mathcal{R}_{\lambda,\mu}$. Such squares are contained in the vertical strip that is deleted in the application of $e_m^\perp$. In other words, these squares are removed by $B(p;u)$ but must not be added afterwards.

- The squares that are contained in both λ and μ, that are both rightmost in their row of λ, and bottommost in their column of μ. Each of these is contained in some of the ν in $\mathcal{R}_{\lambda,\mu}$, but not others. Such squares may have been contained in both a deleted vertical strip and an added horizontal strip or neither. We call the squares in μ with this property λ-*ambiguous*.

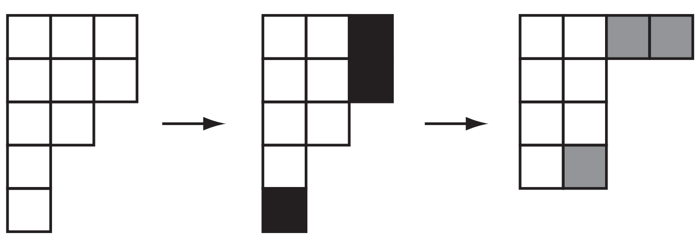

Figure 4.5: The Partition $2^3 1$ in $\mathcal{R}_{3^2 21^2, 42^3}$

Example 4.2.2. *The partition* $2^3 1$ *in* $\mathcal{R}_{3^2 21^2, 42^3}$ *is shown in Figure 4.5. The black squares are those that have been removed and the grey squares are those that have been added. Also, the leftmost partition is* λ*, the rightmost partition is* μ *and the middle partition with the black squares removed is* ν*.*

We can write this sequence of operations in a slightly more compact way as in Figure 4.6.

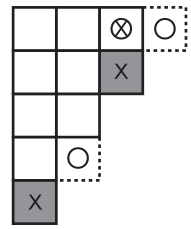

Figure 4.6: The Partition 2^31 in $\mathcal{R}_{3^221^2,42^3}$, Alternate Presentation

Here the squares with an X in them have been removed and the squares with an O in them have been added. Note that squares which contain only an O are of the first type above and those that contain only an X are of the second type. The square which contains both an O and an X is a λ-ambiguous square. Notice, however, that λ-ambiguous squares need not contain both an X and an O in this type of presentation but could also contain no marking. An example of this is the first square in the fourth row.

To make the notion of λ-ambiguous more clear, consider the alternate sequence of operations on 3^221^2. In Figure 4.7 the square in the upper row which was added

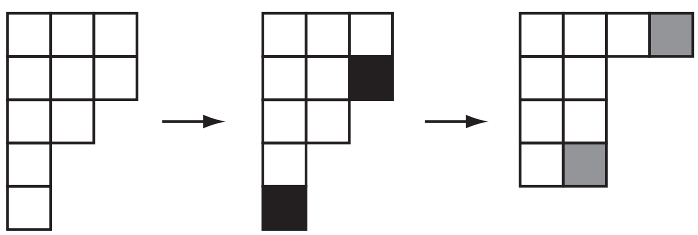

Figure 4.7: The Partition 32^21 in $\mathcal{R}_{3^221^2,42^3}$

and removed in the previous action has now been left untouched. The initial and resulting partitions are still the same, however, which is why we say that such a square is λ-ambiguous. This corresponds to the following alternate presentation. Note that Figure 4.8 has one fewer X than Figure 4.6 and so produces the opposite

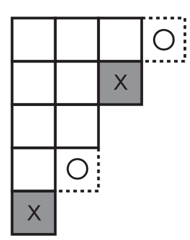

Figure 4.8: The Partition 32^21 in $\mathcal{R}_{3^221^2,42^3}$, Alternate Presentation

sign in (4.11) *for $R_{\lambda,\mu}$. The lemma that follows tells us that this sign change occurs because of the λ-ambiguous square.*

Lemma 4.2.3. *If μ has any λ-ambiguous squares, then*

$$R_{\lambda,\mu} = 0.$$

Proof. We will proceed by constructing a sign reversing involution on the set $\mathcal{R}_{\lambda,\mu}$ which depends on the λ-ambiguous squares in μ.

If μ has any λ-ambiguous squares then let c be the rightmost of these (there is at most one λ-ambiguous square in any column of μ, since it can only occur as the bottommost element of a column). Define the mapping

$$\phi : \mathcal{R}_{\lambda,\mu} \to \mathcal{R}_{\lambda,\mu} : \nu \mapsto \nu'$$

as follows: if ν contains c, then ν' is obtained by removing c from ν; if ν does not contain c, then ν' is obtained by adding c to ν.

We see that the mapping ϕ is well-defined as follows. Since c is rightmost in its row of λ and bottommost in its column of μ, every square of λ in the same column as c and below c must belong to the vertical strip $\lambda - \nu$ and no other squares in this column can belong to this vertical strip, so $\lambda - \nu'$ is a vertical strip whether ν contains c or not. Also, every square of μ in the same row as c and to the right of c must belong to the horizontal strip $\mu - \nu$ and no other squares in this row can belong to this horizontal strip, so $\mu - \nu'$ is a horizontal strip whether ν contains c or not.

Clearly ϕ is an involution on $\mathcal{R}_{\lambda,\mu}$, so it is a bijection, and thus we have

$$R_{\lambda,\mu} = \sum_{\nu \in \mathcal{R}_{\lambda,\mu}} (-1)^{|\lambda|-|\nu|} = - \sum_{\nu' \in \mathcal{R}_{\lambda,\mu}} (-1)^{|\lambda|-|\nu'|} = -R_{\lambda,\mu}, \tag{4.14}$$

where, for the second equality, we have changed the summation variable to $\phi(\nu) = \nu'$. The result follows immediately. □

If $\mathcal{R}_{\lambda,\mu}$ is nonempty and μ has no λ-ambiguous squares, we call μ a λ-*survivor*. In this case there is a unique ν in $\mathcal{R}_{\lambda,\mu}$ and so $R_{\lambda,\mu} = \pm 1$. The terminology is chosen since μ "survives" the involution in Lemma 4.2.3.

Proposition 4.2.4. *In a λ-survivor μ:*

(a) if the rightmost square of row i of λ is not contained in μ, then the rightmost square of row $i-1$ of λ is not contained in μ;

(b) if the bottommost square of column i of μ is not contained in λ, then the bottommost square of column $i-1$ of μ is not contained in λ.

Proof. For part (a), if the rightmost square of row i of λ is not contained in μ and the rightmost square of row $i-1$ of λ is contained in μ, then the latter must be bottommost in its column of μ. But that makes it a λ-ambiguous square, impossible in a λ-survivor. For (b), if the bottommost square of column i of μ is not contained in λ and the bottommost square of column $i-1$ of μ is contained in λ, then the latter is by definition a λ-ambiguous square, impossible in a λ-survivor. □

Now we are able to determine explicitly the action of $B(p;u)$ on a single Schur function s_λ.

Theorem 4.2.5. *For any partition λ, we have*

$$B(p;u)\, s_\lambda = \sum_{i\geq 1} (-1)^{|\lambda|-|\lambda^{(i)}|+i-1} u^{|\lambda^{(i)}|-|\lambda|} s_{\lambda^{(i)}}.$$

Proof. From Proposition 4.2.1 and Lemma 4.2.3, we have

$$B(p;u)\, s_\lambda = \sum_{\mu} R_{\lambda,\mu}\, u^{|\mu|-|\lambda|} s_\mu, \tag{4.15}$$

where the summation is over all λ-survivors μ. Now we characterize the λ-survivors. Suppose $\lambda = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_1 \geq \ldots \geq \lambda_n \geq 1$, and we let $\lambda_0 = \infty$, $\lambda_{n+1} = 0$. Then in a λ-survivor μ, from Proposition 4.2.4(a), the rightmost cells in rows $1, \ldots, j$ of λ are not contained in μ, and the rightmost cells of rows $j+1, \ldots, n$ are contained in μ, for some $j = 0, \ldots, n$ with $\lambda_j > \lambda_{j+1}$. But, in order to avoid the bottommost cell of column λ_{j+1} in μ being λ-ambiguous, then the bottommost cell of column λ_{j+1} in μ cannot be contained in λ. Thus we conclude from Proposition 4.2.4(b) that the bottommost cells in columns $1, \ldots, \lambda_{j+1}$ of μ are not contained in λ. Also, the bottommost squares in columns $\lambda_{j+1}, \ldots, i-1$ of μ are not contained in λ for some $\lambda_{j+1} < i \leq \lambda_j$. Finally, for each $i \geq 1$, there exists a choice of $j = 0, \ldots, n$ for which $\lambda_{j+1} < i \leq \lambda_j$, so the λ-survivor μ described above exists for each $i \geq 1$. This partition μ is obtained from λ by deleting the column strip consisting of the rightmost squares in rows $1, \ldots, j$, and adding the horizontal strip consisting of the bottommost cells in columns $1, \ldots, i-1$. This gives

$$\mu = (\lambda_1 - 1, \ldots, \lambda_j - 1, i-1, \lambda_{j+1}, \ldots, \lambda_n) = \lambda^{(i)},$$

where the last equality comes from the description of $\lambda^{(i)}$ and where $j = u_i(\lambda)$. But we have $R_{\lambda,\lambda^{(i)}} = (-1)^j$, and the result follows from the fact that

$$u_i(\lambda) = |\lambda| - |\lambda^{(i)}| + i - 1$$

and (4.15). $\square$

Note that the right hand side of the above result is a Laurent series in u for each λ, with minimum power of u given by $u^{|\lambda^{(1)}|-|\lambda|} = u^{-l(\lambda)}$.

The following pair of dual corollaries to Theorem 4.2.5 will be particularly convenient for dealing with the KP hierarchy.

Corollary 4.2.6. *For scalars a_λ, $\lambda \in \mathcal{P}$ where $\mathcal{P}$ is the set of partitions, we have*

$$B(p;u) \sum_{\lambda\in\mathcal{P}} a_\lambda s_\lambda = \sum_{\beta\in\mathcal{P}} s_\beta \sum_{k\geq 1} (-1)^{k-1} u^{|\beta|-|\beta^{(-k)}|} a_{\beta^{(-k)}}.$$

Proof. From Theorem 4.2.5 and the equation for $u_i(\lambda)$, we immediately obtain

$$B(p;u)\sum_{\lambda\in\mathcal{P}}a_\lambda s_\lambda=\sum_{\lambda\in\mathcal{P}}a_\lambda\sum_{i\geq 1}(-1)^{u_i(\lambda)}u^{|\lambda^{(i)}|-|\lambda|}s_{\lambda^{(i)}}. \tag{4.16}$$

Now from the code description, it is immediate that $\beta=\lambda^{(i)}$ is equivalent to $\lambda=\beta^{(-k)}$, where $k=u_i(\lambda)+1$. The result follows immediately by changing summation variables in (4.16) from $\lambda\in\mathcal{P}$, $i\geq 1$ to $\beta\in\mathcal{P}$, $k\geq 1$. □

Corollary 4.2.7. *For scalars a_λ, $\lambda\in\mathcal{P}$, we have*

$$B^{\perp}(p;u)\sum_{\lambda\in\mathcal{P}}a_\lambda s_\lambda=\sum_{\alpha\in\mathcal{P}}s_\alpha\sum_{m\geq 1}(-1)^{|\alpha|-|\alpha^{(m)}|+m-1}u^{|\alpha|-|\alpha^{(m)}|}a_{\alpha^{(m)}}.$$

Proof. From Theorem 4.2.5 and the fact that

$$\omega(s_\lambda)=s_{\lambda'}\qquad\text{and}\qquad B^{\perp}(p;u)=\omega B(p;-u)\omega,$$

we obtain

$$B^{\perp}(p;u)\sum_{\lambda\in\mathcal{P}}a_\lambda s_\lambda=\sum_{\lambda\in\mathcal{P}}a_\lambda\omega B(p;-u)\,s_{\lambda'}=\sum_{\lambda\in\mathcal{P}}a_\lambda\sum_{i\geq 1}(-1)^{i-1}u^{|(\lambda')^{(i)}|-|\lambda'|}s_{((\lambda')^{(i)})'}.$$

But, from the code description, it is immediate that $(\lambda')^{(i)}=(\lambda^{(-i)})'$. Since $|\mu'|=|\mu|$ for any partition μ, we can simplify the double summation above to obtain

$$B^{\perp}(p;u)\sum_{\lambda\in\mathcal{P}}a_\lambda s_\lambda=\sum_{\lambda\in\mathcal{P}}a_\lambda\sum_{i\geq 1}(-1)^{i-1}u^{|\lambda^{(-i)}|-|\lambda|}s_{\lambda^{(-i)}}. \tag{4.17}$$

As in the proof of Corollary 4.2.6, we have that $\alpha=\lambda^{(-i)}$ is equivalent to $\lambda=\alpha^{(m)}$, where $i=u_m(\alpha)+1$. The result now follows immediately by changing summation variables in (4.16) from $\lambda\in\mathcal{P}$, $i\geq 1$ to $\alpha\in\mathcal{P}$, $m\geq 1$, and applying

$$u_i(\lambda)=|\lambda|-|\lambda^{(i)}|+i-1$$

to evaluate $u_m(\alpha)$ (which is the exponent of (-1) when the summation is expressed in terms of α, m). □

Among the results in [22] and [33] for Bernstein's operators is

$$B_{\lambda_1}\cdots B_{\lambda_n}1=s_\lambda,$$

where $\lambda=(\lambda_1,\ldots,\lambda_n)$. This result follows immediately from Theorem 4.2.5, together with the fact that $|\lambda^{(i)}|\leq|\lambda^{(i+1)}|$. To compose B_i when they are not ordered as in this result, one simply uses the result that $B_iB_j=-B_{j-1}B_{i+1}$, which follows routinely from Theorem 4.2.5 and considering what happens when two right-steps are switched to up-steps in the two possible orders.

4.3 Plücker Relations and Partition Codes

We consider a set $a_{\mathcal{P}} = \{a_\lambda : \lambda \in \mathcal{P}\}$ of scalars indexed by the set $\mathcal{P}$ of partitions. The *Plücker relations* for $a_{\mathcal{P}}$ are the following system of simultaneous quadratic equations: for all $m \geq 1$ and $\alpha = (\alpha_1, \ldots, \alpha_{m-1}), \beta = (\beta_1, \ldots, \beta_{m+1}) \in \mathcal{P}$ with $l(\alpha) \leq m-1$, $l(\beta) \leq m+1$ (which means that $\alpha_i = 0$ for $i > l(\alpha)$, and $\beta_i = 0$ for $i > l(\beta)$). We have

$$\sum_{k=0}^{m} (-1)^{k-m+1+\ell} a_{(\alpha_1-1,\ldots,\alpha_\ell-1,\beta_{k+1}-k+\ell+1,\alpha_{\ell+1},\ldots,\alpha_{m-1})} \cdot a_{(\beta_1+1,\ldots,\beta_k+1,\beta_{k+2},\ldots,\beta_{m+1})} = 0, \tag{4.18}$$

where $\cdot$ denotes multiplication, $\ell = \ell(k)$ is chosen so that $0 \leq \ell \leq m-1$ and

$$\alpha_\ell - 1 \geq \beta_{k+1} - k + \ell + 1 \geq \alpha_{\ell+1}, \tag{4.19}$$

with the convention that $\alpha_0 = \infty$, $\alpha_m = -\infty$ and so that $\beta_{k+1} - k + \ell + 1 \geq 0$ (if there is no such choice of ℓ, then the term in the summation indexed by k is identically 0). Note that, for each choice of k, ℓ (if it exists) is unique.

In this presentation, each equation is a quadratic alternating summation corresponding to an ordered pair of partitions. Each term in the alternating summation arises from removing a single part from the second partition, and inserting it into the first partition, with some appropriate shift in the remaining parts of both partitions. In our next result we give a different presentation of the Plücker relations, which is more symmetrical in its form, using the notation developed earlier for codes of partitions.

Theorem 4.3.1. *The Plücker relations for $a_{\mathcal{P}}$ are given by the following system of simultaneous quadratic equations: for all $\alpha, \beta \in \mathcal{P}$, we have*

$$\sum_{\substack{i,j \geq 1 \\ |\alpha^{(i)}| + |\beta^{(-j)}| = |\alpha| + |\beta| + 1}} (-1)^{|\alpha| - |\alpha^{(i)}| + i + j} a_{\alpha^{(i)}} \cdot a_{\beta^{(-j)}} = 0.$$

Proof. In the Plücker relations, equation (4.18) is satisfied for each (m, α, β) for $m \geq 1$ and $\alpha, \beta \in \mathcal{P}$ with $l(\alpha) \leq m-1$, $l(\beta) \leq m+1$. Now multiply (4.18) by $(-1)^{m-1}$ to get equation (4.18)' and consider equation (4.18)' for $(m+1, \alpha, \beta)$, where we have $\alpha_m = \beta_{m+2} = 0$. Then, on the left hand side, the term indexed by $k = m+1$ in the latter equation is identically 0 since there is no possible choice of ℓ (to see this, we must have $\beta_{k+1} - k + \ell + 1 \geq 0$, so $\ell \geq m$ and since $0 \leq \ell \leq m$, we must uniquely have $\ell = m$; but then we have $\alpha_\ell - 1 = -1 < 0 = \beta_{k+1} - k + \ell + 1$, contradicting equation (4.19)).

Thus equation (4.18)' for (m, α, β) is identical to equation (4.18)' for $(m+1, \alpha, \beta)$, so there is the following single equation for each $\alpha = (\alpha_1, \alpha_2, \ldots), \beta = (\beta_1, \beta_2, \ldots) \in \mathcal{P}$ (which means that $\alpha_i = 0$ for $i > l(\alpha)$, and $\beta_i = 0$ for $i > l(\beta)$):

$$\sum_{k \geq 0} (-1)^{k+\ell} a_{(\alpha_1-1,\ldots,\alpha_\ell-1,\beta_{k+1}-k+\ell+1,\alpha_{\ell+1},\ldots)} \cdot a_{(\beta_1+1,\ldots,\beta_k+1,\beta_{k+2},\ldots)} = 0, \tag{4.20}$$

where $\ell = \ell(k)$ is chosen so that

$$\alpha_\ell - 1 \geq \beta_{k+1} - k + \ell + 1 \geq \alpha_{\ell+1},$$

with the convention that $\alpha_0 = \infty$.But, from

$$\lambda^{(i)} = (\lambda_1 - 1, \cdots, \lambda_j - 1, i - 1, \lambda_{j+1}, \cdots, \lambda_n)$$

and

$$\lambda^{(-i)} = (\lambda_1 + 1, \cdots, \lambda_{i-1} + 1, \lambda_{i+1}, \cdots),$$

equation (4.20) becomes

$$\sum_{k \geq 0} (-1)^{k+\ell} a_{\alpha^{(\beta_{k+1}-k+\ell+2)}} \cdot a_{\beta^{(-k-1)}} = 0.$$

Finally, note that

$$|\alpha^{(\beta_{k+1}-k+\ell+2)}| + |\beta^{(-k-1)}| = |\alpha| + |\beta| + 1,$$

and the result follows from (4.2) for $u_i(\lambda)$ and the fact that $|\lambda^{(i)}|$ and $|\lambda^{(-i)}|$ are strictly increasing with i as shown in (4.3) and (4.5). □

We now give a few examples of Plücker relations. These examples illustrate that there are redundant equations in the Plücker relations.

Example 4.3.2. *For $\alpha = \beta = (1)$, we obtain $\alpha^{(1)} = \varepsilon$, $\alpha^{(2)} = (1, 1)$, $\alpha^{(3)} = (2, 1)$, and $\beta^{(-1)} = \varepsilon$, $\beta^{(-2)} = (2)$, $\beta^{(-3)} = (2, 1)$, so the corresponding quadratic equation is*

$$-a_\varepsilon \cdot a_{(2,1)} + a_{(2,1)} \cdot a_\varepsilon = 0.$$

But the left hand side of this equation is identically 0, so the equation is redundant.

For $\alpha = \varepsilon$, $\beta = (1, 1, 1)$, we obtain $\alpha^{(1)} = \varepsilon$, $\alpha^{(2)} = (1)$, $\alpha^{(3)} = (2)$, $\alpha^{(4)} = (3)$, and $\beta^{(-1)} = (1, 1)$, $\beta^{(-2)} = (2, 1)$, $\beta^{(-3)} = (2, 2)$, so the corresponding quadratic equation is

$$a_\varepsilon \cdot a_{(2,2)} - a_{(1)} \cdot a_{(2,1)} + a_{(2)} \cdot a_{(1,1)} = 0. \tag{4.21}$$

For $\alpha = (2)$, $\beta = (1)$, we obtain $\alpha^{(1)} = (1)$, $\alpha^{(2)} = (1, 1)$, $\alpha^{(3)} = (2, 2)$, and $\beta^{(-1)} = \varepsilon$, $\beta^{(-2)} = (2)$, $\beta^{(-3)} = (2, 1)$, so the corresponding quadratic equation is

$$-a_{(1)} \cdot a_{(2,1)} + a_{(1,1)} \cdot a_{(2)} + a_{(2,2)} \cdot a_\varepsilon = 0,$$

which is the same equation as (4.21).

4.4 Plücker Relations and Tau Functions

Recall that a symmetric function τ expressed in power sum symmetric functions is a tau function of the KP hierarchy if and only if

$$[u^{-1}]\left(B(p;u)\tau(p)\right)\left(B^{\perp}(\hat{p},u)\tau(\hat{p})\right)=0. \tag{4.22}$$

Using the results above concerning the combinatorial description of the action of B and $B^{\perp}$ on symmetric functions, we give a new proof of the connection between the Schur function coefficients of τ and Plücker relations. Our proof is immediate from Corollaries 4.2.6 and 4.2.7.

Theorem 4.4.1. *Let the coefficient of the Schur function of shape λ in a power series be given by a_λ, $\lambda \in \mathcal{P}$. Then the power series is a τ-function for the KP hierarchy if and only if $a_{\mathcal{P}} = \{a_\lambda : \lambda \in \mathcal{P}\}$ satisfies the Plücker relations.*

Proof. We are given $\tau(p) = \sum_{\lambda\in\mathcal{P}} a_\lambda s_\lambda$ and $\tau(\hat{p}) = \sum_{\lambda\in\mathcal{P}} a_\lambda \hat{s}_\lambda$. Then, from (4.22), it is necessary and sufficient that $a_{\mathcal{P}}$ satisfies $S(p,\hat{p}) = 0$, where

$$S(p,\hat{p}) = [u^{-1}]\left(B(p;u)\sum_{\lambda\in\mathcal{P}} a_\lambda s_\lambda\right)\cdot\left(B^{\perp}(\hat{p};u)\sum_{\mu\in\mathcal{P}} a_\mu \hat{s}_\mu\right).$$

Now, from Corollaries 4.2.6 and 4.2.7, we immediately obtain

$$S(p,\hat{p}) = \sum_{\beta,\alpha\in\mathcal{P}} s_\beta\,\hat{s}_\alpha \sum_{\substack{m,k\geq 1\\ |\alpha^{(m)}|+|\beta^{(-k)}|=|\alpha|+|\beta|+1}} (-1)^{|\alpha|-|\alpha^{(m)}|+m+k} a_{\alpha^{(m)}}\cdot a_{\beta^{(-k)}}.$$

But $S(p,\hat{p}) = 0$ if and only if $[s_\beta\hat{s}_\alpha]S(p,\hat{p}) = 0$ for all $\beta,\alpha\in\mathcal{P}$, since the Schur functions form a basis for symmetric functions, and the result follows immediately from Theorem 4.3.1. $\square$

Often the KP hierarchy is written as a system of simultaneous quadratic partial differential equations for τ. In the next result, we apply Theorem 4.4.1 and the methods of symmetric functions to obtain such a system of partial differential equations, with one equation corresponding to each quadratic equation in the Plücker relations. The result is well known, but we include a simple proof for completeness.

Theorem 4.4.2. *The power series $\tau(p)$ is a τ-function for the KP hierarchy if and only if the following partial differential equation is satisfied for each pair of partitions α and β:*

$$\sum_{\substack{i,j\geq 1\\ |\alpha^{(i)}|+|\beta^{(-j)}|=|\alpha|+|\beta|+1}} (-1)^{|\alpha|-|\alpha^{(i)}|+i+j}\left(s_{\alpha^{(i)}}(p^{\perp})\tau(p)\right)\cdot\left(s_{\beta^{(-j)}}(p^{\perp})\tau(p)\right)=0.$$

(Where, e.g., $s_\lambda(p^{\perp})$ is interpreted as the partial differential operator obtained by substituting $p_n^{\perp}$ for p_n in s_λ for each $n\geq 1$, and using the differential operator form of $p_n^{\perp}$.)

Proof. Let $q = (q_1, q_2, ...)$, where the q_i are independent from the p_j and $\hat{p}_k$. We begin the proof by proving that (I): τ satisfies equation (4.22) if and only if (II): τ satisfies

$$[u^{-1}]\bigg(B(p;u)\tau(p+q)\bigg) \cdot \bigg(B^{\perp}(\hat{p},u)\tau(\hat{p}+q)\bigg) = 0, \tag{4.23}$$

for all q.

It is easy to see that (II) implies (I), by setting $q_i = 0$ for $i \geq 1$.

To prove that (I) implies (II), define the operator $\Theta(p) = \exp\left(\sum_{k\geq 1} q_k \frac{\partial}{\partial p_k}\right)$. Using the multivariate Taylor series expansion of an arbitrary formal power series $f(p)$, we see that

$$f(p+q) = \Theta(p)f(p). \tag{4.24}$$

Also, define operators $\Gamma(p) = \exp\left(\sum_{i\geq 1} \frac{u^i}{i} p_i\right)$ and $\Upsilon(p) = \exp\left(-\sum_{j\geq 1} u^{-j} \frac{\partial}{\partial p_j}\right)$, so that $B(p;u) = \Gamma(p)\Upsilon(p)$. Then we have

$$B(p;u)\tau(p+q) = \Gamma(p)\Upsilon(p)\Theta(p)\tau(p) = \Gamma(p)\Theta(p)\Upsilon(p)\tau(p),$$

from (4.24) and the trivial fact that $\Upsilon(p)$ commutes with $\Theta(p)$. Using (4.24) again, we have the operator identity

$$\Theta(p)\Gamma(p) = \Gamma(p+q)\Theta(p) = \Gamma(q)\Gamma(p)\Theta(p).$$

Combining these expressions and the fact that $\Gamma(q)^{-1} = \Gamma(-q)$ gives

$$B(p;u)\tau(p+q) = \Gamma(-q)\Theta(p)B(p;u)\tau(p).$$

Similarly, we have $B^{\perp}(\hat{p};u) = \Gamma(-\hat{p})\Upsilon(-\hat{p})$, and so obtain

$$B^{\perp}(\hat{p};u)\tau(\hat{p}+q) = \Gamma(q)\Theta(\hat{p})B^{\perp}(\hat{p};u)\tau(\hat{p}).$$

Multiplying these two expressions together, we find that equation (4.23) becomes

$$\Theta(p)\Theta(\hat{p})[u^{-1}]\bigg(B(p;u)\tau(p)\bigg) \cdot \bigg(B^{\perp}(\hat{p};u)\tau(\hat{p})\bigg) = 0,$$

since $\Gamma(-q)\Gamma(q) = 1$, and $\Theta(p)$, $\Theta(\hat{p})$ are independent of t. We conclude that (I) implies (II).

Finally, in order to apply Theorem 4.4.1, we determine the coefficient of the Schur function of shape λ. This gives

$$\begin{aligned}
[s_\lambda(p)]\tau(p+q) &= \langle s_\lambda(p), \tau(p+q)\rangle \\
&= \langle 1, s_\lambda(p^{\perp})\tau(p+q)\rangle \\
&= s_\lambda(p^{\perp})\tau(p+q)\big|_{p=0} \\
&= s_\lambda(q^{\perp})\tau(q),
\end{aligned}$$

and the result then follows from Theorem 4.4.1, replacing q by p. $\square$

As an example of Theorem 4.4.2, we now give one of the quadratic partial differential equations for a τ-function.

Example 4.4.3. *Consider the Plücker equation* (4.21). *Now we have*

$$s_\varepsilon = 1, \quad s_{(1)} = p_1, \quad s_{(2)} = \tfrac{1}{2}(p_1^2 + p_2), \quad s_{(1,1)} = \tfrac{1}{2}(p_1^2 - p_2),$$

$$s_{(2,1)} = \tfrac{1}{3}(p_1^3 - p_3), \quad s_{(2,2)} = \tfrac{1}{12}(p_1^4 - 4p_1p_3 + 3p_2^2),$$

so from Theorem 4.4.2, the partial differential equation for τ that corresponds to (4.21) *is given by*

$$\tfrac{1}{12}\tau\;(\tau_{1111} - 12\tau_{13} + 12\tau_{22}) - \tfrac{1}{3}\tau_1\,(\tau_{111} - 3\tau_3) + \tfrac{1}{4}\,(\tau_{11} + 2\tau_2)\,(\tau_{11} - 2\tau_2) = 0, \quad (4.25)$$

where we use τ_{ijk} to denote $\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}\frac{\partial}{\partial p_k}\tau$, etc.

Often, in the literature of integrable systems, the series $F = \log \tau$ is used instead of τ itself. This series F is often referred to as a *solution* to the KP hierarchy, where the "KP hierarchy" in this context refers to a system of simultaneous partial differential equations for F. Of course, the system of partial differential equations for τ given in Theorem 4.4.2 becomes an equivalent system of partial differential equations for F by substituting $\tau = \exp F$ into the equations of Theorem 4.4.2, and then dividing the equation by the common factor $\exp(2F)$. For example, when we apply this to (4.25), we obtain the equation

$$\tfrac{1}{12}F_{1111} - F_{13} + F_{22} + \tfrac{1}{2}F_{11}^2 = 0,$$

which is an alternate form of the KP equation (1.8).

Chapter 5

Applications

In this final chapter we give a brief account of some combinatorial and geometric applications of the KP hierarchy. Many of the technical details in this chapter have been suppressed and we refer the reader to the literature cited for complete details.

5.1 The Hurwitz Problem

The first application of the KP hierarchy which we will examine is the problem of enumerating equivalence classes of ramified coverings of the sphere with prescribed branching data, also called the Hurwitz problem. We now briefly describe the Hurwitz problem and refer to [20] for more details.

Let X be a genus $g \geq 0$ Riemann surface and let $f : X \to S^2$ be a continuous map from X to the sphere S^2. Let $\{z_1, \cdots z_k\} \subseteq S^2$ be a finite set of points on S^2 and suppose that for some fixed positive integer d and for any $y \in S^2 \backslash \{z_1, \cdots, z_k\}$ there exists some neighborhood V such that $f^{-1}(V)$ is homeomorphic to $V \times S$ where S is a discrete set of size d. We say that f is a *degree d ramified covering* of the sphere with *branch points* $\{z_1, \cdots z_k\}$.

If $f_1 : X_1 \to S^2$ and $f_2 : X_2 \to S^2$ are degree d ramified coverings of S^2 then we say that they are *equivalent* if there exist homeomorphisms

$$\sigma : X_1 \to X_2,$$
$$\rho : S^2 \to S^2$$

such that $\rho f_1 = f_2 \sigma$. The Hurwitz problem is then to enumerate the equivalence classes of degree d ramified covers of S^2, of which there are a finite number.

Suppose $f : X \to S^2$ is a degree d ramified cover of S^2 with branch points $\{z_1, \cdots, z_k\}$. Let D be an open disc such that the branch points are on the boundary of D. There are d connected components in $f^{-1}(D)$ which we label from 1 to d. We call the connected components in the preimage the sheets of the cover and we say that the sheet with the label i is the ith sheet of the cover. If we look at a small neighborhood of z_i, beginning on sheet s and going around z_i counter clockwise, we will arrive at a point on another sheet, say $\pi^{(i)}(s)$. In doing so we construct a permutation $\pi^{(i)}$ for each branch point. If the permutation corresponding to a branch point is a transposition then we say that the branch point is *simple*.

If we begin at some point $x \in S^2\backslash\{z_1, \cdots, z_k\}$ and walk around each branch point as described above, then if we begin on a sheet s, we must end on sheet s. This is because the corresponding loop on $S^2\backslash\{z_1, \cdots, z_k\}$ is contractible to a point. This means that

$$\pi^{(1)} \cdots \pi^{(k)} = 1,$$

where 1 is the identity permutation. This is called the *monodromy condition*. Also, since X is connected, we must be able to move from one sheet to any other and so the subgroup generated by $\pi^{(1)}, \cdots, \pi^{(k)}$ must act transitively. This is called the *transitivity condition*.

A surprising fact about the Hurwitz enumeration problem is that the combinatorial description above completely encodes the cover up to relabeling the sheets (or in other words, up to conjugation of the permutations by a fixed permutation). We thus consider the combinatorial problem of computing the numbers $Cov_d(\lambda^1, \cdots, \lambda^k)$ where each λ^i is a partition of d corresponding to the conjugacy class of $\pi^{(i)}$. We say that the partitions specify the *branching data* of the cover.

The number $Cov_d(\lambda^1, \cdots, \lambda^k)$ is the number of k-tuples, $(\sigma_1, \cdots, \sigma_k) \in S_d^k$, that satisfy the following conditions:

1. $\sigma_i \in \mathcal{C}_{\lambda^i}$ for all i,
2. $\sigma_1 \cdots \sigma_k = 1$ (the monodromy condition),
3. the subgroup generated by $\sigma_1, \cdots, \sigma_k$ is transitive.

We will focus on the special case of *simple Hurwitz numbers* where we allow one of the permutations to be arbitrary and require that the other permutations be transpositions.Note that the monodromy condition and transitivity condition together imply that the transpositions generate a transitive subgroup. The simple Hurwitz numbers are denoted

$$h_k(\alpha) = Cov_d(\alpha, 21^{d-2}, \cdots, 21^{d-2})$$

where $\alpha \vdash d$ and there are k copies of the partition 21^{d-2}. We define the generating function

$$H(z,p) = H(z, p_1, p_2, \cdots) = \sum_{d \geq 1} \frac{1}{d!} \sum_{k=1}^{\infty} \sum_{\alpha \vdash d} h_k(\alpha) \frac{z^k}{k!} p_{\alpha_1} p_{\alpha_2} \cdots .$$

We also define the generating function

$$\widetilde{H} = e^H$$

which is the generating function for possibly disconnected covers. In other words, $\widetilde{H}$ enumerates factorizations in which we don't require that the transpositions generate a transitive subgroup. Note that we don't record the genus of the covering. At this time we do not need this extra piece of information but, as is stated in (5.5), this can be recovered if we need it.

5.2 Join-Cut and the KP Hierarchy

Since the transitivity condition is difficult to work with directly, we will focus on the generating function $\widetilde{H}$. In this case we are counting $(k+1)-$tuples, $(\sigma, \tau_1, \cdots, \tau_k) \in \mathcal{S}_d^{k+1}$ where $\sigma\tau_1 \cdots \tau_k = 1$, the τ_i are transpositions and $\sigma \in \mathcal{C}_\alpha$ for some fixed $\alpha \vdash d$. From the monodromy condition we see that $\tau_1 \cdots \tau_k = \sigma^{-1}$, but since σ^{-1} has the same cycle type as σ, this problem is equivalent to counting factorizations of a permutation in $\mathcal{C}_\alpha$ into transpositions. We now examine what happens to the arbitrary permutation in the factorization problem when we remove a transposition.

Theorem 5.2.1. *The generating function $\widetilde{H}$ is the unique solution to the differential equation*

$$\frac{\partial \widetilde{H}}{\partial z} = J\widetilde{H} \tag{5.1}$$

where

$$J = \frac{1}{2}\sum_{i,j=1}^{\infty}\left((i+j)p_i p_j \frac{\partial}{\partial p_{i+j}} + ijp_{i+j}\frac{\partial^2}{\partial p_i \partial p_j}\right),$$

with initial conditions

$$\widetilde{H}(0, p) = e^{p_1}.$$

Proof. We will sketch the proof here, for more details see [7] and [8].

The left hand side of (5.1) describes the action on $\widetilde{H}$ of removing the first transposition from the factorization. Since a transposition is an involution, we can achieve this by multiplying both sides of the equation $\tau_1 \cdots \tau_k = \sigma$ by τ_1 on the left. This tells us that what results is a factorization of the permutation $\tau_1\sigma$. What we now need to determine is how multiplying by a transposition affects the cycle type.

There are two cases to consider when multiplying a permutation with a transposition, $(l\,m)$. The first is the case when l and m appear in the same cycle of σ. In this case, the action of multiplying by $(l\,m)$ is to split the cycle that contains l and m into two cycles, one of which contains l and the other of which contains m. Given a cycle of length j in σ, there are j choices for the transposition $(l\,m)$ which will split the cycle into two cycles, one of length i and the other of length $j-i$. This is because once we choose l we know that m is the value at distance i from l. Taking into account the fact that the order of the resulting i and $j-i$ cycles doesn't matter, we get (after re-indexing) the term

$$\frac{1}{2}\sum_{i,j=0}^{\infty}(i+j)p_ip_j\frac{\partial}{\partial p_{i+j}}$$

in the operator J.

The second case is when l and m appear in different cycles of σ. In this case the action of multiplying by $(l\,m)$ is to join the two cycles. If the two cycles had lengths i and j then there are i positions in the first cycle and j positions in the second cycle at which we can join the two cycles. In any case, the resulting cycle will have length $i+j$. Again, taking into account symmetry, this gives us the second term,

$$\frac{1}{2}\sum_{i,j=0}^{\infty}ijp_{i+j}\frac{\partial^2}{\partial p_i\partial p_j},$$

in the operator J.

Lastly, to see that the initial conditions hold for the generating function $\widetilde{H}$ we need only notice that $z=0$ implies that there are no transpositions and so the permutation σ must be the identity permutation. □

The operator J in the theorem is often called the *join-cut operator* and equation (5.1) is called the *join-cut equation*.

From the join-cut equation we see that $\widetilde{H}$ can be written as

$$\widetilde{H} = e^{zJ}e^{p_1}. \tag{5.2}$$

We will now show (following [17]) that (5.2) implies that $\widetilde{H}$ is a tau function for the KP hierarchy.

Consider the operator

$$\widehat{J} = \frac{1}{6}\sum_{i,j=-\infty}^{\infty} :\Lambda_i\Lambda_j\Lambda_{-(i+j)}:$$

on the fermionic Fock space where $:\Lambda_{i_1}\cdots\Lambda_{i_k}: = \Lambda_{\pi(i_1)}\cdots\Lambda_{\pi(i_k)}$ and π is a permutation of the indices such that $\pi(i_1)\le\cdots\le\pi(i_k)$. This is called the *normal ordering* of the operators $\Lambda_{i_1},\cdots,\Lambda_{i_k}$.

It is clear that the operator $\widehat{J}$ is an element of the algebra $\overline{g\ell}_\infty$ since the index of the shift operator which acts first will always be positive and so will become the zero operator for a large enough index. This implies that $e^{z\widehat{J}}$ acts as an operator in $\overline{GL}_\infty$. Similarly, $\Lambda_{-1}\in\overline{g\ell}_\infty$ and so $e^{\Lambda_{-1}}\in\overline{GL}_\infty$. Notice that we are considering operators in $\overline{GL}_\infty$ rather than GL_∞ which have been our focus up until now. The results in Chapters 2 and 3 still hold only we end up with formal power series solutions to the KP hierarchy rather than polynomial solutions. We now know that the function

$$\widehat{H} = e^{z\widehat{J}} e^{\Lambda_{-1}} \psi_0$$

is in the $\overline{GL_\infty}$ orbit of ψ_0. Now, let σ be the algebraic isomorphism in the Boson-Fermion correspondence and let ρ be the algebraic isomorphism from the bosonic Fock space to the ring of symmetric functions (via the change of basis (4.6) to the scaled power sum symmetric functions). We see that

$$\begin{aligned}
\rho\sigma(\widehat{J}) &= \rho\left(\frac{1}{2}\sum_{i,j=1}^{\infty}\left(ijx_ix_j\frac{\partial}{\partial x_{i+j}} + (i+j)x_{i+j}\frac{\partial^2}{\partial x_i \partial x_j}\right)\right) \\
&= J, \\
\rho\sigma(\Lambda_{-1}) &= \rho(x_1) = p_1, \\
\rho\sigma(\psi_0) &= \rho(1) = 1,
\end{aligned}$$

where we have used (4.6) and (3.15). This implies that

$$\rho\sigma(\widehat{H}) = \widetilde{H}$$

so that $\widetilde{H}$ is a tau function for the KP hierarchy in the variables p_i/i.

5.3 ELSV and Hodge Integrals

We will now briefly describe *Hodge integrals* and how they can be related to the Hurwitz problem discussed in the previous section. We are able to relate the computation of Hodge integrals with the Hurwitz numbers by making use of a formula, originally introduced by T. Ekadahl, S. Lando, M. Shapiro and A. Vainshtein, called the ELSV formula. For more on Hodge integrals we refer to [6].

Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne-Mumford compactification of the moduli space of genus g curves with n marked points. If we let $X = (\mathcal{C}, x_1, x_2, \cdots, x_n)$ be a point in $\overline{\mathcal{M}}_{g,n}$ where $\mathcal{C}$ is a genus g curve and $x_1, \cdots, x_n$ are the marked points then we can associate with each marked point the line bundle $\mathcal{L}_i$ whose fiber at the point X is the cotangent line to $\mathcal{C}$ at x_i. We let ψ_i be the first Chern class of $\mathcal{L}_i$. Each $\overline{\mathcal{M}}_{g,n}$ also admits a natural rank g vector bundle $\mathbb{E}$, the Hodge bundle, whose fiber at X corresponds to the space of global differentials on X. We let λ_k be the kth Chern class of $\mathbb{E}$ and we let $\lambda_0 = 1$.

We denote the intersection numbers (also called Hodge integrals) by

$$\langle \lambda_k \tau_{m_1} \cdots \tau_{m_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \lambda_k \psi_1^{m_1} \cdots \psi_n^{m_n}.$$

Note that since the intersection numbers are independent of the order of the ψ classes, we may write the intersection numbers using exponential notation so that $\tau_i^{m_i}$ denotes m_i copies of τ_i.

Following Kazarian [17], we define

$$\widetilde{G}(u; T_0, T_1, \cdots) = \sum_{j,k_0,k_1,\cdots} (-1)^j \langle \lambda_j \tau_0^{k_0} \tau_1^{k_1} \cdots \rangle u^{2j} \frac{T_0^{k_0}}{k_0!} \frac{T_1^{k_1}}{k_1!} \cdots ,$$

and we denote by $G(u; q_1, q_2, \cdots)$ the series obtained from $\widetilde{G}$ by the linear substitution of variables defined by the recursion

$$T_{k+1} = \sum_{m \geq 1} m(u^2 q_m + 2uq_{m+1} + q_{m+2}) \frac{\partial}{\partial q_m} T_k \tag{5.3}$$

with $T_0 = q_1$.

The *ELSV formula* states that

$$\frac{h_k(\alpha)}{k!} = \prod_{i=1}^{\ell(\alpha)} \frac{\alpha_i^{\alpha_i}}{\alpha_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{1 - \lambda_1 + \lambda_2 - \cdots \pm \lambda_g}{\prod_{i=1}^{\ell(\alpha)} (1 - \alpha_i \psi_i)} \tag{5.4}$$

where the genus is given by the *Riemann-Hurwitz formula*:

$$k = 2g - 2 + \ell(\alpha) + |\alpha|. \tag{5.5}$$

It can then be shown, using (5.4) (see [17]), that Hurwitz series H and the generating function G are related by a change of variables. In particular, suppose we have two variables, x and y, related by

$$x = \frac{y}{1 + zy} e^{-\frac{zy}{1+zy}},$$

$$y = \sum_{b \geq 1} \frac{b^b}{b!} z^{b-1} x^b$$

where the indeterminate z is the same as that used in the generating function H. Note that x and y are inverse to one another with respect to the Lagrange inversion formula [10]. We now construct a change of variables from p_b (used in H) to q_k (used in G) by

$$p_b = \sum_{k \geq b} c_k^b z^{k-b} q_k \tag{5.6}$$

where the rational coefficients c_k^b are determined by the expansion

$$x^b = \sum_{k \geq b} c_k^b z^{k-b} y^k.$$

Also, let

$$H_1 = \sum_{b=1}^{\infty} \frac{b^{b-2}}{b!} p_b z^{b-1},$$

$$H_2 = \frac{1}{2} \sum_{b_1,b_2=1}^{\infty} \frac{b_1^{b_1} b_2^{b_2}}{(b+1) b_1! b_2!} p_{b_1} p_{b_2} z^{b_1+b_2}.$$

Theorem 5.3.1. *Under the change of variables* (5.6), *we have*

$$(H - H_1 - H_2) = G(z^{1/3}; z^{4/3}q_1, z^{8/3}q_2, z^{12/3}q_3, \cdots).$$

Proof. See Kazarian [17]. □

The reason for removing H_1 and H_2 is that the corresponding moduli spaces, $\overline{\mathcal{M}}_{0,1}$ and $\overline{\mathcal{M}}_{0,2}$, do not exist.

It can also be shown that the change of basis (5.6) is an automorphism of the KP hierarchy so that the following result is true:

Theorem 5.3.2. *The generating function $e^{G(u;q_1,q_2,\cdots)}$ is a tau function for the KP hierarchy in the variables q_i/i.*

Proof. See Kazarian [17]. □

5.4 Witten's Conjecture and the KdV Hierarchy

We now show how Witten's conjecture follows from the results in the previous section. Recall that Witten's conjecture states that the generating function

$$F(t_0, t_1, \cdots) = \left\langle \exp\left(\sum_i t_i\tau_i\right)\right\rangle$$

is such that e^F is a tau function for the KdV hierarchy.

The *KdV hierarchy* can be formed from the KP hierarchy by adding the constraint that the function is free of all even indexed parameters. In other words, τ is a *tau function* for the KdV hierarchy if and only if it is a tau function for the KP hierarchy and

$$\frac{\partial \tau}{\partial t_{2n}} = 0, \qquad \forall n \geq 1.$$

Example 5.4.1. *We know that the simplest equation in the KP hierarchy is the KP equation* (1.8)*:*

$$\frac{3}{4}\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} - \frac{3}{2}u\frac{\partial u}{\partial x} - \frac{1}{4}\frac{\partial^3 u}{\partial x^3}\right),$$

where $t = x_1, y = x_2$ and $x = x_3$. If we add the additional constraint that $\frac{\partial u}{\partial y} = 0$ then we are left with

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} - \frac{3}{2}u\frac{\partial u}{\partial x} - \frac{1}{4}\frac{\partial^3 u}{\partial x^3}\right) = 0$$

which, after integrating with respect to x once, gives us the KdV equation (1.1) *:*

$$\frac{\partial u}{\partial t} - \frac{3}{2}u\frac{\partial u}{\partial x} - \frac{1}{4}\frac{\partial^3 u}{\partial x^3} = 0. \tag{5.7}$$

Note that (5.7) *is the same equation as* (1.1) *and that they are related by a rescaling of the variables.*

Recall that the recursion for the variables T_k in $\widetilde{G}$ is:

$$T_{k+1} = \sum_{m \geq 1} m(u^2 q_m + 2u q_{m+1} + q_{m+2}) \frac{\partial}{\partial q_m} T_k,$$

so that

$$T_k|_{u=0} = (2k-1)!! q_{2k+1}.$$

Here we have used the double factorial to mean $(2k-1)!! = (2k-1)(2k-3)\cdots(1)$. Also,

$$G(0; q_1, q_2, \cdots) = \sum_{k_0, k_1, \cdots} \langle \tau_0^{k_0} \tau_1^{k_1} \cdots \rangle \frac{(T_0|_{u=0})^{k_0}}{k_0!} \frac{(T_1|_{u=0})^{k_1}}{k_1!} \cdots,$$

so we see that

$$F(t_0, t_1, \cdots) = G(0; t_0, 0, t_1, 0, \frac{t_2}{3!!}, 0, \frac{t_3}{5!!}, 0, \cdots)$$

where on the right hand side we have $q_{2d+1} = \frac{t_d}{(2d-1)!!}$ and $q_{2d} = 0$. Then, since e^G is a tau function for the KP hierarchy, we get immediately that e^F is a tau function for the KdV hierarchy.

5.5 The Double Hurwitz Problem

We now turn to the problem of computing the double Hurwitz numbers. The double Hurwitz numbers are similar to the Hurwitz numbers encountered previously except we allow two permutations of arbitrary type. In particular, for $\alpha, \beta \vdash d$, we let

$$d_k(\alpha, \beta) = Cov_d(\alpha, \beta, (21^{d-2}), \cdots, (21^{d-2}))$$

be the double Hurwitz numbers where here there are k copies of the partition (21^{d-2}). We also let

$$D = \sum_{\substack{d,k \\ \alpha,\beta \vdash d}} \frac{1}{d!} \frac{z^k}{k!} d_k(\alpha, \beta) p_\alpha q_\beta,$$

$$\widetilde{D} = e^D = \sum_{\substack{d,k \\ \alpha,\beta \vdash d}} \frac{1}{d!} \frac{z^k}{k!} \widetilde{d}_k(\alpha, \beta) p_\alpha q_\beta$$

be the corresponding generating functions. Here we write p_α for the power sum symmetric function indexed by α and q_β for the power sum symmetric function indexed by β in variables distinct from those in p_α. Note also that the $\widetilde{d}_k(\alpha, \beta)$ are the disconnected double Hurwitz numbers, i.e., the number of solutions to the equation

$$\sigma \rho \tau_1 \cdots \tau_k = 1$$

where $\sigma \in \mathcal{C}_\alpha, \rho \in \mathcal{C}_\beta$ and $\tau_i \in \mathcal{C}_{21^{d-2}}$ and there is no transitivity requirement.

One of the reasons for changing our point of view to that of the disconnected Hurwitz numbers is that we can rewrite the question of computing the disconnected double Hurwitz numbers as a question about the center of the group algebra $\mathbb{C}\mathcal{S}_d$. This means that we can construct the generating function $\widetilde{D}$ using representation theory.

Theorem 5.5.1 (Frobenius' formula)**.** *The number of solutions to the equation*

$$\tau_1 \cdots \tau_k = 1$$

in $\mathcal{S}_d$, where $\tau_i \in \mathcal{C}_{\lambda^{(i)}}$ with $\lambda^{(i)} \vdash d$, is equal to

$$\frac{|\mathcal{C}_{\lambda^{(1)}}| \cdots |\mathcal{C}_{\lambda^{(k)}}|}{|\mathcal{S}_d|} \sum_{\mu \vdash d} \frac{\chi^\mu_{\lambda^{(1)}} \cdots \chi^\mu_{\lambda^{(k)}}}{(\chi^\mu_{1^d})^{k-2}}.$$

Proof. A proof can be found in the appendix of [20] in the more general context of an arbitrary finite group. $\square$

In particular, we see that:

$$\begin{aligned}
\widetilde{d}_k(\alpha, \beta) &= \frac{|\mathcal{C}_\alpha||\mathcal{C}_\beta||\mathcal{C}_{21^{d-2}}|^k}{d!} \sum_{\mu \vdash d} \frac{\chi^\mu_\alpha \chi^\mu_\beta (\chi^\mu_{21^{d-2}})^k}{(\chi^\mu_{1^d})^{k-2}} \\
&= \sum_{\lambda \vdash d} \frac{(\dim \lambda)^2}{d!} f_\alpha(\lambda) f_\beta(\lambda) (f_{21^{d-2}}(\lambda))^k,
\end{aligned}$$

where $\dim \lambda = \chi^\lambda_{1^d}$, and

$$f_\lambda(\mu) = |\mathcal{C}_\lambda| \frac{\chi^\mu_\lambda}{\dim \lambda}.$$

For convenience we will also write $f_2 = f_{21^{d-2}}$.

Using the fact that

$$\begin{aligned}
s_\lambda(p) &= \frac{1}{d!} \sum_{\mu \vdash d} \chi^\lambda_\mu |\mathcal{C}_\mu| p_\mu \\
&= \frac{\dim \lambda}{d!} \sum_{\mu \vdash d} f_\mu(\lambda) p_\mu,
\end{aligned}$$

we can rewrite the generating function $\widetilde{D}$ as

$$\begin{aligned}
\widetilde{D} &= \sum_{\substack{d,k\\ \alpha,\beta\vdash d}} \frac{1}{d!}\frac{z^k}{k!}\widetilde{d}_k(\alpha,\beta)p_\alpha q_\beta \\
&= \sum_{\substack{d,k\\ \alpha,\beta\vdash d}} \frac{1}{d!}\frac{z^k}{k!}p_\alpha q_\beta \sum_{\lambda\vdash d}\frac{(\dim\lambda)^2}{d!} f_\alpha(\lambda)f_\beta(\lambda)(f_2(\lambda))^k \\
&= \sum_{\substack{d,k\\ \alpha,\beta\vdash d}} \frac{z^k}{k!}(f_2(\lambda))^k \left(\frac{\dim\lambda}{d!}\sum_{\alpha\vdash d} f_\alpha(\lambda)p_\alpha\right)\left(\frac{\dim\lambda}{d!}\sum_{\beta\vdash d} f_\beta(\lambda)q_\beta\right) \\
&= \sum_{d,\lambda\vdash d} s_\lambda(p)s_\lambda(q)\left(\sum_k \frac{z^k}{k!}(f_2(\lambda))^k\right) \\
&= \sum_\lambda s_\lambda(p)s_\lambda(q)e^{zf_2(\lambda)}.
\end{aligned}$$

Our next step is to write this generating function as an element in the orbit $\overline{GL_\infty}\psi_0$.

First we mention that $f_2(\lambda)$ can be expressed as a polynomial in the parts of λ (see [22]):

$$\begin{aligned}
f_2(\lambda) &= \frac{1}{2}\sum_i \left[(\lambda_i - i + \frac{1}{2})^2 - (-i+\frac{1}{2})^2\right] \\
&= \sum_{k\in S(\lambda)_+}\frac{k^2}{2} - \sum_{k\in S(\lambda)_-}\frac{k^2}{2},
\end{aligned}$$

where $S(\lambda) = \{\lambda_i - i + \frac{1}{2}, i\in\mathbb{Z}\}$, $S_+ = S\backslash(\mathbb{Z}_{\le 0} - \frac{1}{2})$ and $S_- = (\mathbb{Z}_{\le 0} - \frac{1}{2})\backslash S$.

Define the operator

$$F_2 = \sum_{k>1}\frac{(k-\frac{1}{2})^2}{2}\hat{v}_i\check{v}_i^* - \sum_{k\le 0}\frac{(k-\frac{1}{2})^2}{2}\check{v}_i^*\hat{v}_i.$$

We see immediately (using the combinatorial description of the wedging and contracting operators) that

$$F_2 v_\lambda = f_2(\lambda)v_\lambda.$$

If we also define the operator

$$T = \exp\left(\sum_{n\ge 1}\frac{q_n}{n}\Lambda_{-n}\right)$$

then we find that (see the proof of theorem 3.2.2)

$$T\psi_0 = \sum_\lambda s_\lambda(q)v_\lambda.$$

Putting these two operators together we see that

$$\begin{aligned} e^{zF_2} T\psi_0 &= \sum_\lambda s_\lambda(q) e^{zF_2} v_\lambda \\ &= \sum_\lambda s_\lambda(q) e^{zf_2(\lambda)} v_\lambda, \end{aligned}$$

so that

$$\rho\sigma\left(e^{zF_2} T\psi_0\right) = \widetilde{D}.$$

Now, since T and e^{zF_2} are operators in $\overline{GL_\infty}$, this tells us immediately that $\widetilde{D}$ is a tau function for the KP hierarchy.

This technique was originally used in [25] to prove the more general result that $\widetilde{D}$ is a tau function for the Toda hierarchy.

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