Chromatic Number of Random Signed Graphs

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

We naturally extend Bollobás's classical method and result about the chromatic number of random graphs $\chi(G(n,p)) \sim n/\log_b n$ (for p constant, $b = 1/(1-p)$) [6] to the chromatic number of random signed graphs to obtain $\chi(G(n,p,q)) \sim n/\log_b n$ (for p constant, $b = 1/(1-p)$, $q = o(1)$). In the process, we will give a sufficient condition for the type of graph structures on which this method is applicable. We also give a sufficient bound on q under which a.a.s. the chromatic number of $G(n,p,q)$ is unchanged before and after adding negative edges.

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Dedication

To my parents, who, by coming to Canada, sacrificed their careers for a chance at a better life for me.

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Chapter 1

Introduction

1.1 History

The problem of graph colouring, or how many colours are needed to colour the vertices of a graph so that no two adjacent vertices share a colour, has a long history. Its oldest and arguably most well-known version, the Four-Colour Problem, originally posed by de Morgan to his friend Hamilton in 1852, asks whether every planar map can be coloured in four colours. It puzzled mathematicians for more than a century before being proven true by Appel and Haken in 1976 [3]. Directly or indirectly thanks to the four-colour problem, graph colouring has become one of the cornerstones of graph theory.

It is obvious that no fewer colours should be needed to colour a graph than any of its subgraphs. However, there are many infinite collections of graphs with small chromatic numbers (like bipartite graphs). One may argue that this is due to their edges being carefully chosen. A natural question, then, would be: "How would the chromatic number of a uniformly random graph grow based on its number of vertices?" A more general version of question was first answered by Bollobás in 1988 [6], based on a model of *random graphs*.

Questions related to the "typical" behaviour of graphs, such as the above, pushed a number of mathematicians to independently start studying random graphs in the late 1950s. The main model that they used is now commonly known as the *Erdős-Rényi* model. Those two were the first to exploit its probabilistic properties, establishing what we now know as the probabilistic method [7]. Erdős and Rényi's original model in 1959 was $G(n, m)$, where every graph on n (labelled) vertices and m edges appears with equal probability [10]. What we now call the Erdős-Rényi model can also denote $G(n, p)$, where every edge is present in

the graph independently with probability p. In other words, for a given graph G_0 with n vertices and m edges,

$$\mathbf{P}\{G(n,p) = G_0\} = p^m(1-p)^{\binom{n}{2}-m}.$$

In particular, when $p = \frac{1}{2}$, $G(n,p)$ is uniform over the set of graphs on n vertices. This allows the properties of uniformly random graphs to be studied, which is exactly what Bollobás did for their chromatic number [6]. Indeed, he obtained that, for $0 < p < 1$ constant, the chromatic number of $G(n,p)$ is a.a.s. (asymptotically almost surely, meaning the ratio of the two sides of the following expression converges to 1 in probability)

$$\chi(G(n,p)) \sim \frac{n}{2\log_{\frac{1}{1-p}} n}.$$

Our thesis is deeply related to Bollobás's analysis and result. We will be looking at a particular generalization of graph colouring which stems from a generalization on the concept of a graph itself. Graphs can be seen as representing a network of objects with relationships between them. For example, in the colouring problem for maps, the graph can be seen as representing adjacency of entities on the map. However, adjacency does not indicate whether the relationship is positive or negative, in for example a social network. This is precisely why Harary, in 1953 [13], introduced *signed graphs*. Signed graphs are simply graphs where each edge is labelled positive or negative, which are supposed to represent the type of relationship between its endpoints. As many concepts in graph theory are extended to signed graphs, Zaslavsky, in 1982 [20], gives the first extension of chromatic number to signed graphs, based on the chromatic polynomial (which is a polynomial $p_G(x)$, such that $p_G(n)$ outputs the number of colourings of G with n colours, if $n \in \mathbb{N}$).

Since then, there have been a number of different definitions of signed colourings [16] [18], because it is not obvious how should negative edges exclude colours when the total number of colours is non-even. However, a common theme is that, when the number of colour used *is* even, all definitions correspond to a colouring which maps the vertices to $\{-k, \ldots, -1, 1, \ldots, k\}$ for a $2k$-colouring, where positive edges forbid same-coloured endpoints, and negative edges forbid opposite-colour endpoints. Since the odd numbers do not affect the asymptotic behaviour, we will be restricting signed colourings to only using an even number of colours.

Finally, we need a model of random signed graphs. In 2012, El Maftouhi et al. [17] defined $G(n,p,q)$ random signed graphs and gave results on their balance. This is very similar to the $G(n,p)$ model, but here p denotes the probability that two vertices are positively adjacent, and q the probability that they are negatively adjacent.

This naturally extends the Erdős-Rényi model, under which the asymptotic behaviour of the chromatic number has a known result. Therefore, $G(n,p,q)$, after a slight modification, is the model that we will be using to study the asymptotic behaviour of the chromatic number of random signed graphs, establishing analogies with $G(n,p)$ along the way.

Our thesis naturally extends long-established methods and results onto signed graphs, with some practical applications. In general, random signed graphs form an infinite testing ground for theories and algorithms about signed graphs. In our case, knowing the asymptotic behaviour of random signed graphs can help test the accuracy of approximation algorithms for signed colouring. We also give some ideas of applications for signed colouring itself in 2.2.1.

1.1.1 Outline

The thesis will roughly be divided as follows. In particular, the main work is in Chapter 3 and in Section 4.2.

In Chapter 2, we will give the necessary background on random graphs (Section 2.1), probability theory (Section 2.3), and signed graphs (Section 2.2). In particular, we recommend all readers to have a look at Subsection 2.1.2, where we give some standard results on random graphs, since the results and the discussions there will be important for the main results of the thesis.

In Chapter 3, we will introduce the $G(n,p,q)$ model of random signed graphs, define a signed analogue to independent sets and derive its expectation (Section 3.1), and show that the signed chromatic number is unchanged for small q (Section 3.2). The model was first introduced by El Maftouhi et al [17], while the rest of the chapter is original work.

In Chapter 4, we will first cover Bollobás's use of martingales in finding the chromatic number of $G(n,p)$ random graphs [6], generalized as a *method of martingales* (Section 4.1, then apply it to the signed chromatic number of signed random graphs (Section 4.2), to obtain the main result Thereom 4.2.3:

Theorem (Chromatic Number of $G(n,p,q)$)**.** *Let $0 < p = p(n) < 1$ be asymptotically bounded away from $0, 1$, $q = o(1)$, and $b = \frac{1}{1-p}$. Then there exist functions $b_1 = b_1(n)$ and $b_2 = b_2(n)$, such that $b_1 \sim b_2 \sim b$, and a.a.s.*

$$\frac{n}{2\log_{b_1} n} \leq \chi(G(n,p,q)) \leq \frac{n}{2\log_{b_2} n}.$$

The generalization of the method and application to signed graphs are original work.

Finally, in Chapter 5, we summarize the thesis and give some open problems for future research (Section 5.1).

Chapter 2

Background Tools

We will start by giving some background on the standard methods and results relevant to our work. They mainly fall into three areas, namely random (unsigned) graphs, (non-random) signed graphs, and probability theory. Since we shall make extensive use of standard notation for probability theory in this thesis, we point to [5] and [19] for basic definitions and notations.

2.1 Random Graphs

The contents of this section are well-known in the study of random graphs. We point to [1] and [7] for reference and further reading. Defined formally, the Erdős-Rényi model that we will consider, $G(n,p)$, is as follows.

Definition 2.1.1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space consisting of $\binom{n}{2}$ i.i.d. (independent and identically distributed) $\mathbf{Ber}(p)$ random variables $\{X_e\}_{e\in E(K_n)}$. Then $G(n,p)$, mapping from $\Omega = \{0,1\}^{E(K_n)}$ to the set of graphs on n vertices, is such that for $e \in E(K_n)$,

$$\begin{cases} e \in E(G(n,p)), & X_e = 1; \\ e \notin E(G(n,p)), & \text{else.} \end{cases}$$

In other words, we flip a coin independently for every edge of K_n to decide whether to add that edge to $G(n,p)$, with a probability p of being added.

2.1.1 Standard Tools

In studying random graphs, one of the most common tasks is to investigate the behaviour of some graph property of $G(n,p)$ as $n \to \infty$. Ideally, we would like to be able to make statements about the *probability* of $G(n,p)$ containing some properties, such as connectedness, having a perfect matching, being Hamiltonian, etc., as $n \to \infty$, but this probability is often hard to compute, due to dependences between the events involved. Expectations, on the other hand, are often easy to compute due to linearity. This motivates the *first-moment method*, one of the most basic and common approaches to random graph problems.

Proposition 2.1.2 (First-Moment Method)**.** *Suppose a sequence of non-negative random variables $\{X_n\}_{n\geq 1}$ is such that*

$$\lim_{n\to\infty} \mathbf{E}X_n = 0.$$

Then, for all fixed $\epsilon > 0$, $X_n \to 0$ in probability, meaning

$$\lim_{n\to\infty} \mathbf{P}\{X_n \geq \epsilon\} = 0.$$

In particular, if $X_n \in \mathbb{N}$, then $\lim_{n\to\infty} \mathbf{P}\{X_n = 0\} = 1$.

This is a direct corollary of a result from probability theory. Intuitively, non-negativity coupled with an expectation going to 0 "squeezes" the sequence to 0.

Proposition 2.1.3 (Markov's inequality)**.** *Let X be a non-negative random variable. Then, for all $c > 0$,*

$$\mathbf{P}\{X \geq c\} \leq \frac{\mathbf{E}X}{c}.$$

Proof. Write

$$\mathbf{E}X = \mathbf{E}\{X \cdot \mathbf{1}_{X\geq c}\} + \mathbf{E}\{X \cdot \mathbf{1}_{X<c}\}.$$

Note $\mathbf{E}\{X \cdot \mathbf{1}_{X\geq c}\} \geq \mathbf{E}\{c\mathbf{1}_{X\geq c}\} = c\mathbf{P}\{X \geq c\}$, while $\mathbf{E}\{X \cdot \mathbf{1}_{X<c}\} \geq 0$. Thus

$$\mathbf{E}X \geq c\mathbf{P}\{X \geq c\} \implies \frac{\mathbf{E}X}{c} \geq \mathbf{P}\{X \geq c\}. \qquad \square$$

Proof of First-moment Method. For every n, by Markov's inequality

$$\mathbf{P}\{X_n \geq \epsilon\} \leq \frac{\mathbf{E}X_n}{\epsilon}.$$

Since ϵ is constant,

$$\lim_{n\to\infty} \mathbf{P}\{X_n \geq \epsilon\} \leq \frac{1}{\epsilon} \lim_{n\to\infty} \mathbf{E}X_n = 0. \qquad \square$$

The first moment method shows that the expectations of the absolute values going to 0 force the sequence to converge to 0 *in probability*, a much stronger statement. A natural question would be whether the expectations going to ∞ forces the sequence to be large with high probability. This turns out to be false.

Example 2.1.4. Consider a sequence of random variables $\{X_n\}_{n\geq 1}$ where X_n is n^2 with probability $\frac{1}{n}$ and 0 with probability 0. Then $\mathbf{E}X_n = n \to \infty$, but X_n actually converges to 0 in probability.

Before we explain how to deal with this situation, let us first introduce the *Landau notation*, which is a standard notation for analysing asymptotics and which we will use extensively throughout the thesis.

Definition 2.1.5. Let $f(n)$ and $g(n)$ be two functions from $\mathbb{N}$ to $\mathbb{R}$. We say

1. $f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{|f(n)|}{|g(n)|} = 0$;
2. $f(n) = O(g(n))$ if $\limsup_{n\to\infty} \frac{|f(n)|}{|g(n)|} < \infty$;
3. $f(n) \sim g(n)$ if $\lim_{n\to\infty} \frac{|f(n)|}{|g(n)|} = 1$;
4. $f(n) = \Omega(g(n))$ if $\liminf_{n\to\infty} \frac{|f(n)|}{|g(n)|} > 0$;
5. $f(n) = \omega(g(n))$ if $\lim_{n\to\infty} \frac{|f(n)|}{|g(n)|} = \infty$;
6. $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

In particular, a function is $o(1)$ if it is asymptotically 0, it is $\omega(1)$ if it is asymptotically ∞, and $f(n) \sim g(n)$ if $f(n) = (1 + o(1))g(n)$ (in other words, $f(n) - g(n) = o(g(n))$). In that case, $f(n)$ and $g(n)$ are *asymptotically the same*. Furthermore, note that the equality symbol, whenever Landau notation is involved, is *not* symmetric.

Let us now go back to dealing with how to bound a sequence of random variables *away* from 0. If the second moments $\mathbf{E}\{X_n^2\}$ of the sequence (which is no smaller than the square of the first moment) turn out to be asymptotically the same as the squares of the first moments $(\mathbf{E}X_n)^2$, and both go to infinity, then it turns out that the sequence itself will also follow its expectation to infinity, at the same "rate".

Proposition 2.1.6 (Second-Moment Method)**.** *Suppose a sequence of random variables $\{X_n\}_{n\geq 1}$ is such that the expectation of each X_n is defined, and furthermore*

$$\mathbf{E}\{X_n^2\} = (1 + o(1))(\mathbf{E}X_n)^2 = \omega(1).$$

Then there exists a sequence $\{\epsilon_n\}_{n\geq 0}$, such that $\epsilon_n = o(\mathbf{E}X_n)$, and

$$\lim_{n\to\infty} \mathbf{P}\{|X_n - \mathbf{E}X_n| \geq \epsilon_n\} = 0.$$

This is another direct corollary of a result from probability theory. Intuitively, a smaller variance forces the random variable to be around its expectation, and $\mathbf{E}\{X_n^2\} \sim (\mathbf{E}X_n)^2$ makes the variance small (indeed, the variance, being their difference, is $o((\mathbf{E}X_n)^2)$).

Proposition 2.1.7 (Chebyshev's nequality)**.** *Let X be a random variable whose expectation is defined. Let $\sigma = \sqrt{\mathrm{var}(X)\}}$ be the standard deviation of X. Then, for all $c > 0$,*

$$\mathbf{P}\{|X - \mathbf{E}X| \geq c\sigma\} \leq \frac{1}{c^2}.$$

Proof. Applying Markov's inequality Proposition 2.1.3 to $(X - \mathbf{E}X)^2$ and $c^2\sigma^2$, we get

$$\mathbf{P}\{|X - \mathbf{E}X| \geq c\sigma\} = \mathbf{P}\{(X - \mathbf{E}X)^2 \geq c^2\sigma^2\} \leq \frac{\mathbf{E}\{(X - \mathbf{E}X)^2\}}{c^2\sigma^2} = \frac{1}{c^2}. \qquad \square$$

Proof of Second-Moment Method. Let σ_n denote the standard deviations of $\mathbf{E}X_n$. Without loss of generality $\sigma_n \neq 0$ for all n. For any sequence $\{\omega_n\}_{n\geq 1}$ non-zero such that $\omega_n \to \infty$ as $n \to \infty$, we have by Chebyshev's inequality

$$\mathbf{P}\{|X_n - \mathbf{E}X_n| \geq \omega_n \sigma_n\} \leq \frac{1}{\omega_n^2} \to 0.$$

Choose $\omega_n = \sqrt{\frac{|\mathbf{E}X_n|}{\sigma_n}}$. We first show

$$\lim_{n\to\infty} \sqrt{\frac{|\mathbf{E}X_n|}{\sigma_n}} = \infty.$$

Indeed

$$\frac{(\mathbf{E}X_n)^2}{\sigma_n^2} = \frac{(\mathbf{E}X_n)^2}{\mathbf{E}\{X_n^2\} - (\mathbf{E}X_n)^2} = \frac{(\mathbf{E}X_n)^2}{o(1)(\mathbf{E}X_n)^2} = \omega(1).$$

Now, it suffices to confirm

$$\epsilon_n := \omega_n \sigma_n = \sqrt{|\mathbf{E}X_n|\sigma_n} = \mathbf{E}X_n \sqrt{\frac{\sigma_n}{|\mathbf{E}X_n|}} = o(\mathbf{E}X_n),$$

which is true as $\sqrt{\frac{|\mathbf{E}X_n|}{\sigma_n}} = \omega(1)$. □

Those two methods focused on the limiting *moments* of random variables, instead of directly working with limiting probabilities. Although more difficult, it is not *impossible* do the latter, by actually computing the *limiting distribution*. One way to achieve this is, again, through moments. This is known as the *method of moments*.

Theorem 2.1.8 (Method of Moments)**.** *Let $\{X_n\}_{n\geq 0}$ be a sequence of random variables. Let X be a random variable with distribution μ. Suppose all their moments exist,*

$$\lim_{n\to\infty} \mathbf{E}\{X_n^r\} = \mathbf{E}X^r \tag{2.1.1}$$

for all $r \geq 1$, and furthermore suppose that μ is the unique probability measure with its sequence of moments (in other words, if Y is a random variable and $\mathbf{E}\{X^r\} = \mathbf{E}\{Y^r\}$ for all $r \geq 1$, then $X \sim Y$). Then $X_n \xrightarrow{\mathcal{D}} X$.

In particular, (2.1.1) holds if and only if

$$\lim_{n\to\infty} \mathbf{E}\{(X_n)_r\} = \mathbf{E}(X)_r$$

for all $r \geq 1$ where $(x)_r = x(x-1)\cdots(x-r+1) = \frac{\Gamma(x+1)}{\Gamma(x-r+1)}$ is the falling factorial (if $x \geq r$ is non-negative integer then $(x)_r$ is the number of r-permutations in a set of x elements).

Sketch of Proof. This is Theorem 30.2 from [5]. Because the proof in [5] uses some preliminary results, we will cite those results without proof and so only give a sketch here.

Given $\lim_{n\to\infty} \mathbf{E}\{X_n^2\}$ is convergent, the set $\{\mathbf{E}\{X_n^2\}\}_{n\geq 0}$ is bounded by some constant $K > 0$, so by Markov's inequality Proposition 2.1.3, for all $\epsilon > 0$,

$$\mathbf{P}\{|X_n| > \frac{K}{\epsilon}\} \leq \frac{\mathbf{E}\{X_n\}}{K/\epsilon} \leq \epsilon.$$

Let μ_n denote the distribution of X_n for each $n \geq 0$. The above implies that the sequence of distributions $\{\mu_n\}_{n\geq 0}$ is *tight*. This means that every subsequence of $\{\mu_n\}$ has a convergent subsubsequence that converges weakly (Theorem 25.10 from [5]).

For contradiction suppose $\{X_n\}$ does not converge in distribution to X. In other words, there exists $x \in \mathbb{R}$ such that $\mu(\{x\}) = 0$ (i.e. $\mathbf{P}\{X < x\} = \mathbf{P}\{X \leq x\}$), but there exists a subsequence n_k where

$$\lim_{k\to\infty} |\mu_{n_k}((-\infty, x]) - \mu((-\infty, x])| > 0.$$

By Theorem 25.10 from [5] $\{\mu_{n_k}\}$ has a weakly convergent subsubsequence $\{\mu_{n_{k_\ell}}\}$ to some distribution ν. Let Y have distribution ν. Due to the choice of $\{\mu_{n_k}\}$, $\nu \neq \mu$, however for all $r \geq 1$

$$\mathbf{E}\{X^r\} = \lim_{\ell\to\infty} \mathbf{E}\{X_{n_{k_\ell}}^r\} = \mathbf{E}\{Y^r\}.$$

But this means $X \sim Y$ since the moments of X uniquely determine the distribution of X, a contradiction. □

$\mathbf{E}(X)_r$ is more straightforward to compute than $\mathbf{E}X^r$ particularly when $X = \sum_k X_k$ for some Bernoulli random variables $\{X_k\}$. In that case, as by definition $(X)_r$ counts the number of r-permutations in the set $\{X_k : X_k = 1\}$, we have

$$(X)_r = \sum_{(k_1,\ldots,k_r) \text{ distinct}} \prod_{j=1}^{r} X_{k_j} \implies \mathbf{E}(X)_r = \sum_{(k_1,\ldots,k_r) \text{ distinct}} \mathbf{E}\left\{\prod_{j=1}^{r} X_{k_j}\right\}.$$

A very nice note about the method of moments is that whether X is the unique distribution with its sequence of moments depends solely on X, and this would work regardless of the sequence of random variables whose convergence we are trying to determine. Thus it is easy to identify distributions which are known to work as limiting distributions.

Proposition 2.1.9. *Given $\lambda > 0$, the Poisson distribution* $\mathbf{Pois}(\lambda)$ *is the unique distribution with its moments.*

Sketch of Proof. This results stems from Theorem 30.1 from [5] which ultimately comes from the fact that characteristic functions characterize distributions. Theorem 30.1 from [5] states that, if X is a random variable such that $\mathbf{E}\{|X|^r\} < \infty$ for all $r \geq 1$, and

$$\sum_{r=0}^{\infty} \frac{\mathbf{E}\{X^r\}t^r}{r!} < \infty \tag{2.1.2}$$

for $t \in (-\epsilon, \epsilon)$ where $\epsilon > 0$, then the distribution of X is the unique distribution with its sequence of moments. In particular, (2.1.2) holds wherever the moment generating function $M_X(t) = \mathbf{E}\{e^{tX}\}$ is analytic, as its Maclaurin series is (2.1.2).

Now let us consider $X \sim \mathbf{Pois}(\lambda)$. Recall for $k \geq 0$,

$$\mathbf{P}\{X = k\} = \frac{\lambda^k e^{-\lambda}}{k!}.$$

This means for $t \in \mathbb{R}$,

$$M_X(t) = \mathbf{E}\{e^{tX}\} = \sum_{k=0}^{\infty} \frac{e^{tk}\lambda^k e^{-\lambda}}{k!} = e^{-\lambda}e^{\lambda e^t}$$

which is clearly analytic for all $t \in \mathbb{R}$. □

Corollary 2.1.10. *Let $\{X_n\}_{n\geq 0}$ be a sequence of random variables and suppose, for all $r \geq 1$,*

$$\lim_{n\to\infty} \mathbf{E}\{(X_n)_r\} = \lambda^r$$

for some $\lambda > 0$. Then $X_n \xrightarrow{\mathcal{D}} \mathbf{Pois}(\lambda)$.

Proof. By the method of moments Theorem 2.1.8 and the fact that $\mathbf{Pois}(\lambda)$ is the unique distribution with its moments Proposition 2.1.9, it suffices to show that for $X \sim \mathbf{Pois}(\lambda)$, $\mathbf{E}(X)_r = \lambda^r$. There are many ways to derive this result; the following gives it some intuition.

It is known that $\mathbf{Bin}(n, \lambda/n) \Rightarrow \mathbf{Pois}(\lambda)$ as $n \to \infty$. Let $X_n \sim \mathbf{Bin}(n, \lambda/n)$ and $X \sim \mathbf{Pois}(\lambda)$. Fix $r \geq 1$. Then

$$\lim_{n\to\infty} \mathbf{E}\{(X_n)_r\} = \mathbf{E}\{(X)_r\}.$$

It is known that $X_n = \sum_{k=1}^{n} Y_k$, where Y_k are i.i.d. $\mathbf{Ber}(\lambda/n)$ (the number of successes after n independent trials, each with probability λ/n). Then

$$\begin{aligned}\mathbf{E}\{(X_n)_r\} &= \mathbf{E}\left\{\sum_{(k_1,\ldots,k_r) \text{ distinct}} \prod_{j=1}^{r} Y_{k_j}\right\} = \sum_{(k_1,\ldots,k_r) \text{ distinct}} \prod_{j=1}^{r} (\lambda/n) \\ &= \lambda^r \cdot \frac{n(n-1)\cdots(n-r+1)}{n^r},\end{aligned}$$

and clearly $\lim_{n\to\infty} \frac{n(n-1)\cdots(n-r+1)}{n^r} = 1$. □

The Poisson distribution is commonly encountered as the limiting distribution for sequences of random variables related to random graphs, partly because of its link to binomial distributions. When X_n are sums of identical Bernoulli random variables whose dependence becomes negligible as $n \to \infty$, which often happens as the graph becomes large, one may expect $\{X_n\}$ to converge to a Poisson distribution.

We will now use the above methods to prove some standard results about $G(n, p)$ relevant to this thesis.

2.1.2 Standard Results

In general, Friedgut and Gallai in 1996 [11] obtained that if a property about $G(n, p)$ is *monotonic*, meaning any supergraph of a graph with that property also has it (by working with negations, this also works with properties closed under subgraph), then that property will have a *threshold* $p_0(n)$, meaning that $G(n, p_n)$ will have that property a.a.s. if $p_n \ggg p_0(n)$, and $G(n, p_n)$ will a.a.s. not have that property if $p_n \lll p_0(n)$. In other words, as one moves p from 0 to 1, properties will "pop into existence" as p crosses thresholds. This idea of thresholds is omnipresent in random graphs. The two results that we present below are threshold results.

The first result, due to Bollobás and Erdős from 1976 for fixed $0 < p < 1$ [8], will be about the independence number of $G(n,p)$ (which we will denote by $\alpha(G(n,p))$), for *any* p. More precisely, we will compute the *expected number* of independent sets of size r for r inside a range of values. Then, the first moment method can easily obtain an upper bound on the independence number (by observing that the expected number of independent sets goes to 0 for larger r). Similarly, one could use the second moment method to obtain a lower bound on the independence number, but we will actually obtain a much stronger result later.

For r possibly a function of n, let $E'(n,r)$ denote the expected number of independent sets of size r in $G(n,p)$. Then

$$E'(n,r) = \binom{n}{r}(1-p)^{\binom{r}{2}}.$$

The result is that, as long as p is asymptotically away from $0, 1$, by varying r by a constant, one can change $E'(n,r)$ from asymptotically 0, to a constant power of n. The former would bound $\alpha(G(n,p))$ from above a.a.s. using the first moment method.

Proposition 2.1.11. *Let $0 < p = p(n) < 1$ be asymptotically bounded away from $0, 1$ (i.e. $\limsup p < 1$ and $\liminf p > 0$) and let $b := \frac{1}{1-p}$. Suppose that $c = c(n) > 0$ is such that $c = o(n)$. Let*

$$r = r(n) = 2\log_b(ecn/(2\log_b n)).$$

If there exists a constant $C_1 \in \mathbb{R}$ such that $\limsup(1 - \frac{2\log c}{\log b}) \leq C_1$, then, for each $\epsilon > 0$ constant,

$$E'(n,r) = o(n^{C_1+\epsilon}).$$

If there exists a constant $C_2 \in \mathbb{R}$ such that $\liminf(1 - \frac{2\log c}{\log b}) \geq C_2$, then, for each $\epsilon > 0$ constant,

$$E'(n,r) = \omega(n^{C_2-\epsilon}).$$

Note that since

$$2\log_b(ecn/(2\log_b n)) = 2\log_b n - 2\log_b\log_b n + 2\log_b(e/2) + 2\log_b c,$$

in the case where c is a constant (which is what we will usually have), modifying c changes r additively by a constant amount and $r \sim 2\log_b n$.

Furthermore, note that $\log b$ is asymptotically bounded away from $0, \infty$ due to p being asymptotically bounded away from $0, 1$. As $c \to \infty$, $1 - \frac{2\log c}{\log b} \to -\infty$ so $E'(n,r) = o(1)$ for some choice of constant c (more precisely, any $c > \sqrt{b}$ would work). On the other hand, as $c \to 0$, $1 - \frac{2\log c}{\log b} \to \infty$ so $E'(n,r)$ is polynomial in n for some choice of constant c (more precisely, any $c < b^{\frac{1-\gamma}{2}}$ gives $E'(n,r) = \omega(n^\gamma)$).

Therefore, by varying c by a constant amount, one can choose C_1, C_2 to be arbitrarily large or small (negative) constants, adjusting $E'(n,r)$ from asymptotically 0 to polynomial in n, as long as p is asymptotically away from $0, 1$. The fact that Proposition 2.1.11 allows for the weaker assumption on p being asymptotically bounded away from $0, 1$ (as opposed to p being constant) is the main reason why we can keep using this weaker assumption throughout the thesis.

Proof. Note that $c = o(n)$ means $r(n) \sim 2\log_b n = O(\sqrt{n})$, so $\binom{n}{r} \sim \frac{n^r}{r!}$. Thus

$$\begin{aligned}
E'(n,r) &= \binom{n}{r}(1-p)^{\binom{r}{2}} \\
&= (1-o(1))\frac{n^r}{r!}\exp\left\{\frac{\log(1-p)r(r-1)}{2}\right\} \\
&= (1-o(1))\frac{n^r}{r!}\exp\left\{-(r-1)\log(ecn/(2\log_b n))\right\} \\
&= (1-o(1))\frac{(en)^r}{r^r\sqrt{2\pi r}}\left(\frac{ecn}{2\log_b n}\right)^{-(r-1)} \\
&= (1-o(1))\frac{en}{r\sqrt{2\pi r}}\left(\frac{2\log_b n}{cr}\right)^{r-1}
\end{aligned}$$

Now note (recall $2\log_b n \sim r$)

$$\begin{aligned}
\left(\frac{2\log_b n}{cr}\right)^{r-1} &= \exp\left\{(r-1)\log\left(\frac{2\log_b n}{cr}\right)\right\} = \exp\left\{\frac{2\log n}{\log b}(1-o(1))\log\left(\frac{1+o(1)}{c}\right)\right\} \\
&= n^{\frac{2}{\log b}(-\log c + \log(1+o(1)))(1-o(1))} = n^{-\frac{2\log c}{\log b}(1+o(1))}
\end{aligned}$$

so

$$E'(n,r) = (1-o(1))\frac{e}{\sqrt{2\pi}\cdot 2^{\frac{3}{2}}}\cdot\frac{n^{1-\frac{2\log c}{\log b}(1+o(1))}}{(\log_b n)^{3/2}}.$$

In particular, if there exists $C_1 \in \mathbb{R}$ such that $\limsup(1 - \frac{2\log c}{\log b}) \le C_1$, then $E'(n,r) = o(n^{C_1+\epsilon})$. If there exists $C_2 \in \mathbb{R}$ such that $\liminf(1 - \frac{2\log c}{\log b}) \ge C_2$, then $E'(n,r) = \omega(n^{C_2-\epsilon})$. □

Recall that for any $\epsilon > 0$, choosing $c = \sqrt{b} + \epsilon$ gives $E'(n, r) = o(1)$, which means by the first moment method Proposition 2.1.2, $G(n, p)$ a.a.s. contains no independent set of size

$$\begin{aligned} &2 \log_b \left(\frac{en(\sqrt{b} + \epsilon)}{2 \log_b n} \right) \\ &= 2 \log_b n - 2 \log_b \log_b n + 2 \log_b \left(\frac{e(\sqrt{b} + \epsilon)}{2} \right) \end{aligned}$$

for any fixed ϵ. Thus a.a.s. each colour class has size less than $2 \log_b n - 2 \log_b \log_b n + 2 \log_b(e\sqrt{b}/2)$, so a.a.s.

$$\chi(G(n, p)) \geq \frac{n}{2 \log_b n - 2 \log_b \log_b n + 2 \log_b(e\sqrt{b}/2)}.$$

We will later see that it is possible to asymptotically achieve this lower bound. In other words, we can partition "almost" every vertex of $G(n, p)$ into colour classes with size close to $\alpha(G(n, p))$.

For the second result, we will give an outline on proving the threshold for *perfect matchings* (we will only consider even n, which is equivalent to a matching covering all but at most one vertex). The strategy is as follows.

We first take an arbitrary bipartite subgraph of $G(2n, p)$ with parts of size n, $G(n, n, p)$, by conditioning $G(2n, p)$ to be a subgraph of $K_{n,n}$. We will first show that for some p_n, the limiting distribution of the number of isolated vertices is Poisson, for parameters spanning $\lambda > 0$. Then, we will outline a proof for the fact that for p_n in that range, isolated vertices are the only thing preventing $G(n, n, p)$ from having a perfect matching.

Proposition 2.1.12. *Let $p = p(n) = \frac{\log n + c}{n}$ where c is constant. Let X_n denote the number of isolated vertices in $G(n, p)$ and let $X \sim \mathbf{Pois}(2e^{-c})$. Then $X_n \xrightarrow{\mathcal{D}} X$.*

Proof. For $1 \leq k \leq 2n$, let $Y_{n,k}$ denote the indicator on whether the k-th vertex of $G(n, n, p)$

is an isolated vertex. Then $X_n = \sum_{k=1}^{2n} Y_{n,k}$. Fix $r \geq 1$,

$$\begin{aligned}
\mathbf{E}\{(X_n)_r\} &= \sum_{(k_1,\ldots,k_r) \text{ distinct}} \mathbf{E}\left\{\prod_{j=1}^{r} Y_{n,k_j}\right\} \\
&= \sum_{\ell=0}^{r} \sum_{(k_1,\ldots,k_r) \text{ with } \ell \text{ in first part}} \mathbf{P}\{k_1,\ldots,k_r \text{ isolated}\} \\
&= \sum_{\ell=0}^{r} \binom{n}{\ell}\binom{n}{r-\ell} r!(1-p)^{n\ell+n(r-\ell)-\ell(r-\ell)} \\
&= \sum_{\ell=0}^{r} \frac{n^r r!}{\ell!(r-\ell)!}(1-o(1))(1-p)^{nr-O(1)} \\
&= (1-o(1))n^r \exp\{(nr-O(1))(-p+O(p^2))\} \sum_{\ell=0}^{r} \binom{r}{\ell} \\
&= (1-o(1))n^r 2^r \exp\{-r(\log n + c) + O(p)\} = (1-o(1))n^{r-r}(2e^{-c})^r (n^{\frac{1}{n}})^{O(1)} \\
&\sim (2e^{-c})^r = \mathbf{E}\{(X)_r\}
\end{aligned}$$

so by Corollary 2.1.10 $X_n \xrightarrow{\mathcal{D}} X$. □

This is an example of a sum of dependent Bernoulli random variables (each vertex being isolated or not) whose "dependency" becomes negligible as $n \to \infty$. Each vertex had a probability

$$(1-p)^{n-1} = \left(1 - \frac{\log n + c}{n}\right)^{n-1} = \exp\left\{-\log n - c + O((\log n/n)^2)\right\} \sim \frac{2e^{-c}}{2n}$$

to be isolated, so if these were actual independent Bernoulli then by the proof of Corollary 2.1.10 their sum should also converge to $\mathbf{Pois}(2e^{-c})$.

Recall that a characterization of a graph G with (A,B)-perfect matchings ($|A| = |B|$ are a bipartition of $V(G)$) is that neither A nor B contains S with fewer neighbours than $|S|$. This is the *Hall's theorem*. The following is a consequence of it.

Proposition 2.1.13. *Suppose G (A,B)-bipartite with $|A| = |B| = n$ fails Hall's condition. Then there exists $S \subseteq A$ or $S \subseteq B$ such that*

- $|N(S)| = |S| - 1$,
- $|S| \leq \lceil \frac{n}{2} \rceil$, *and*
- *every vertex in* $N(S)$ *has at least two neighbours in* S.

Proof. We will prove that taking the smallest S (least $|S|$) that fails Hall's condition works.

- If $|S| > |N(S)| + 1$ then one can remove a vertex from S: let $v \in S$, then

$$|N(S \setminus \{v\})| \leq |N(S)| < |S| - 1 = |S \setminus \{v\}|$$

  so $S \setminus \{v\}$ still fails Hall's condition.

- Assume $|N(S)| = |S| - 1$ and without loss of generality $S \subseteq A$. If $|S| > \lceil \frac{n}{2} \rceil$ then $|N(S)| \geq \lceil \frac{n}{2} \rceil$, so $|B \setminus N(S)| = n - |N(S)| = n - |S| + 1 < \frac{n}{2} \leq |S|$, but its neighbours cannot be in S, i.e.

$$N(B \setminus N(S)) \subseteq A \setminus S \implies |N(B \setminus N(S))| \leq n - |S| < |B \setminus N(S)|$$

  so $B \setminus N(S)$ still fails Hall's condition and is smaller than S.

- If $v \in N(S)$ has only one neighbour $u \in S$ then $S \setminus \{u\}$ has at most

$$|N(S) \setminus \{v\}| = |N(S)| - 1 < |S| - 1 = |S \setminus \{u\}|$$

  neighbours so $S \setminus \{u\}$ still fails Hall's condition. □

Now, if $|S| = 1$ then it is an isolated vertex. It can be shown that for p_n in the right range (which contains $\frac{\log n + c}{n}$ for c constant), a.a.s. S cannot exist for $|S| \geq 2$. This would prove that the threshold for perfect matchings is $\frac{\log n}{n}$. In fact, if $p_n \geq \frac{c \log n}{n}$ for $c > 1$ then $G(n, p_n)$ has a perfect matching a.a.s. and if $p_n \leq \frac{c \log n}{n}$ for $c < 1$ then $G(n, p_n)$ a.a.s. has no perfect matching (since it would contain isolated vertices). Such a threshold is known as a *sharp threshold*. Unlike thresholds, sharp thresholds are known to not always exist even for monotonic properties.

Lemma 2.1.14. *For $p = p(n) = \frac{\log n + c}{n}$ where c is constant, a.a.s. $G(n, n, p)$ contains no S as described in Proposition 2.1.13 of size 2 or more.*

Proof. Let A denote the first part and B the second part of $G(n, n, p)$. When $|S| = 2$, S has one neighbour so $S, N(S)$ form what we call a *cherry*. We expect to have

$$\begin{aligned}
&2 \sum_{x_1, x_2 \in A} \sum_{y \in B} \mathbf{P}\{N(\{x_1, x_2\}) = \{y\}\} \\
&= 2\binom{n}{2} np^2 (1-p)^{2(n-1)} \\
&\leq n^3 p^2 \exp\{-p(2n-2)\} = n^3 p^2 \exp\{-2(\log n + c) + 2p\} \\
&= n^{3-2} p^2 e^{-2c} \exp\{o(1)\} = \frac{(\log n)^2}{n} e^{-2c}(1 + o(1)) = o(1)
\end{aligned}$$

cherries, so by the first moment method Proposition 2.1.2 a.a.s. there are no cherries.

Now let X_k denote the number of sets S from Proposition 2.1.13 of size $k \geq 3$. We will show

$$\mathbf{E}\{\sum_{k=3}^{\lceil \frac{n}{2} \rceil} X_k\} = o(1)$$

which will show by the first moment method that a.a.s. there are no S of size 3 or more. Note

$$\mathbf{E}\{X_k\} \leq 2\binom{n}{k}\binom{n}{k-1}\left(\binom{k}{2}p^2\right)^{k-1}(1-p)^{k(n-k+1)}$$

where 2 is from interchanging A, B; $\binom{n}{k} \leq (\frac{en}{k})^k$ from the number of choices of $S \subseteq A$; $\binom{n}{k-1} \leq (\frac{en}{k-1})^{k-1}$ from $N(S) \subseteq B$; $\binom{k}{2}p^2 \leq \frac{k^2}{2}p^2$ from the fact that every $v \in N(S)$ has at least two neighbours in S (this is the only inequality); and $(1-p)^{k(n-k+1)} \leq \exp\{-(n-k+1)pk\} \leq \exp\{-npk/2\}$ (recall $k \leq \lceil n/2 \rceil$) from the fact that S must have

no neighbours outside $N(S)$. Then

$$\begin{aligned}
\mathbf{E}\{X_k\} &\leq 2\left(\frac{en}{k}\right)^k \left(\frac{en}{k-1}\right)^{k-1} \left(\frac{k^2}{2}\right)^{k-1} p^{2(k-1)} \exp\{-(n-k+1)pk\} \\
&\leq \frac{2en}{k}\left(\frac{k^2}{k(k-1)}\right)^{k-1} \left(\frac{en(\log n + c)}{2n}\right)^{2(k-1)} \exp\{-\frac{n}{2}pk\} \\
&\leq \frac{2en}{k}\left(1+\frac{1}{k-1}\right)^{k-1} (e(\log n + c))^{2(k-1)} \exp\left\{-\frac{\log n}{2}k\right\} \\
&\leq 2e^2 n \left(\frac{(e(\log n + c))^2}{n^{\frac{1}{2}}}\right)^k
\end{aligned}$$

The sum of geometric series with $o(1)$ ratios is asymptotically the first term, so

$$\sum_{k=3}^{\infty}\left(\frac{(e(\log n + c))^2}{n^{\frac{1}{2}}}\right)^k = \left(\frac{(e(\log n + c))^2}{n^{\frac{1}{2}}}\right)^3 /(1-o(1)) = o(1/n),$$

so

$$\sum_{k=3}^{\lceil \frac{n}{2} \rceil} \mathbf{E}\{X_k\} \leq 2e^2 n \sum_{k=3}^{\infty}\left(\frac{(e(\log n + c))^2}{n^{\frac{1}{2}}}\right)^k = o(1). \qquad \square$$

Corollary 2.1.15 (Threshold for Perfect Matching)**.** *Let $p = p(n)$ and let ω_n be an arbitrary sequence going to infinity. $G(2n,p)$ a.a.s. contains a perfect matching if $p > \frac{\log n + \omega_n}{n}$, and $G(n,p)$ a.a.s. does not contain a perfect matching if $p < \frac{\log n - \omega_n}{n}$. In particular, $\frac{\log n}{n}$ is a sharp threshold for perfect matchings.*

Proof. Let $c \in \mathbb{R}$ be fixed and let $p = p(n) = \frac{\log n + c}{n}$. Trivially

$$\mathbf{P}\{G(2n,p) \text{ contains a PM}\} \leq \mathbf{P}\{G(2n,p) \text{ contains no isolated vertices}\}$$

Fix an arbitrary n vertices in $V(G)$ and let $G(n,n,p)$ be the random bipartite subgraph of $G(2n,p)$ induced by that bipartition. Then by Proposition 2.1.13 and Lemma 2.1.14

$$\begin{aligned}
\mathbf{P}\{G(2n,p) \text{ contains a PM}\} &\geq \mathbf{P}\{G(n,n,p) \text{ contains a PM}\} \\
&= \mathbf{P}\{G(n,n,p) \text{ contains no isolated vertices}\} - o(1) \\
&\geq \mathbf{P}\{G(2n,p) \text{ contains no isolated vertices}\} - o(1)
\end{aligned}$$

Therefore, by Proposition 2.1.9

$$\begin{aligned}\lim_{n\to\infty} \mathbf{P}\{G(2n,p) \text{ contains a PM}\} &= \lim_{n\to\infty} \mathbf{P}\{G(2n,p) \text{ contains no isolated vertices}\} \\ &= \mathbf{P}\{\mathbf{Pois}(2e^{-c}) = 0\} = e^{-2e^{-c}}.\end{aligned}$$

Now, let $p' > \frac{\log n + \omega_n}{n}$. Since $G(2n,p)$ containing a perfect matching is a monotonic property, by embedding $G(2n,p)$ as a subgraph of $G(2n,p')$ for each n, which can be done for $\omega_n > c$ (which is eventually true),

$$\lim_{n\to\infty} \mathbf{P}\{G(2n,p') \text{ contains a PM}\} \geq \lim_{n\to\infty} \mathbf{P}\{G(2n,p) \text{ contains a PM}\} = e^{-2e^{-c}}$$

and this is true for any $c \in \mathbb{R}$. By choosing $c \to \infty$ we can make $e^{-2e^{-c}} \to 1$.

Similarly, let $p' < \frac{\log n - \omega_n}{n}$. By embedding $G(2n,p')$ as a subgraph of $G(2n,p)$ for each n, which can again be done for $-\omega_n < c$ (which is also eventually true),

$$\lim_{n\to\infty} \mathbf{P}\{G(2n,p') \text{ contains a PM}\} \leq \lim_{n\to\infty} \mathbf{P}\{G(2n,p) \text{ contains a PM}\} = e^{-2e^{-c}}$$

□

We will later (in Section 3.2) make use of this threshold on perfect matchings to study when does the signed chromatic number stay *unchanged* from the unsigned chromatic number (of its positive subgraph).

2.2 Signed Graphs

Signed graphs were first introduced by Harary in 1953 [13] to represent notions of like and dislike in social networks. Most of the results in this introductory part are due to him.

Definition 2.2.1. A *signed graph* G is an unsigned *underlying graph*, denoted $\underline{G} = (V(G), E(G))$, labelled with an edge sign

$$\sigma : E(G) \to \{-1, 1\}.$$

We use $E^+(G)$ to denote the set of positive edges, $E^-(G)$ the set of negative edges, $G^+ = (V(G), E^+(G))$ the *positive subgraph*, and $G^- = (V(G), E^-(G))$ the *negative subgraph*.

One can imagine the vertices to be people, positive edges to represent friendship and negative edges to represent enemies. The foremost property exclusive to signed graphs, studied by Harary, is that of *balance*. The idea is that if one thinks of friendship as a transitive property, of friends of enemies as enemies, and of enemies of enemies as friends, then a friendship network is balanced if and only if nobody is both somebody's friend and enemy.

Definition 2.2.2. Given a signed graph G, the sign of a multiset of edges S is defined to be

$$\sigma(S) = \prod_{e \in S} \sigma(e)$$

the product of the signs of the edges in S.

We say G is *balanced* if all walks in G have their sign determined by their endpoints.

The fundamental result obtained by Harary in [13], intuitively, states that a social network is balanced if and only if the people can be divided into two camps, where two people in the same camp can never be enemies, and two people in opposite camps can never be friends. Furthermore, this is if and only if the network is precisely a network of only friends, but with some friends "flipped" into enemies. This latter notion of "flipping" is formalized as follows.

Definition 2.2.3. Given a signed graph G and $v \in V(G)$, a *vertex switch* at v, denoted $h_v(G)$, is a signed graph with the same vertex and edge sets as G, but

$$\sigma_{h_v(G)}(e) = \begin{cases} \sigma_G(e), & e \text{ not incident to } v; \\ -\sigma_G(e), & e \text{ incident to } v. \end{cases}$$

We say two graphs are *switch equivalent* if one can be obtained from a sequence of vertex switches on the other.

One should imagine a switched vertex as a friend who suddenly decided to join the enemy's camp, so all of their former friends are now enemies and all former enemies are now friends.

Theorem 2.2.4 (Fundamental Theorem of Signed Graphs [13]). *For a signed graph G, the following are equivalent.*

1. *G is balanced.*
2. *All paths in G have their sign determined by their endpoints.*
3. *There exists $H \subseteq V(G)$, such that $G[H], G[V \setminus H]$ both contain no negative edges, while there are no positive edges with one end in H and the other outside.*
4. *G is switch equivalent to $\underline{G}$ labelled all-positive.*

Proof. 1 $\implies$ 2: Paths are walks.

2 $\implies$ 3: First assume that G is connected. Fix $v \in V(G)$. Let H be the set of vertices in G reachable from v via a positive path. Then for $uw \in E(G[H])$, we can prove by induction on $d = \min\{d(v,u), d(v,w)\}$ that $\sigma(uw) = 1$.

If $d = 0$ then $v \in \{u, w\}$, so there exists a positive path between u, w. By 2, as uw is another path between u, w, $\sigma(uw) = 1$.

If $d > 0$ then without loss of generality suppose $d(v,u) = d$, achieved by some path P. Then w is not reached by P since else $d(v,w) \leq d-1$, so $P \cup \{uw\}$ is a path. By IH P is positive, and by 2 $P \cup \{uw\}$ is positive so $\sigma(uw) = 1$.

If G is not connected then apply 2 to every component of G and take H to be the union of the H from every component.

3 $\implies$ 4: We show G can be obtained by G', where G' is $\underline{G}$ relabelled all-positive, by vertex switching every vertex in H once in any order. Call the obtained graph $h_H(G)$. Indeed,

$$\sigma_{h_H(G)}(e) = \begin{cases} \sigma_{G'}(e) = 1, & \text{its endpoints are switched a total even number of times;} \\ -\sigma_{G'}(e) = -1, & \text{its endpoints are switched a total odd number of times.} \end{cases}$$

If $e \in G[H]$, then both endpoints are switched once, so $\sigma_{h_H(G)}(e) = 1$; if $e \in G[V \setminus H]$, then both endpoints are not switched, so $\sigma_{h_H(G)}(e) = 1$; if $e \in \delta(H)$, then one endpoint is switched once while is other is not switched, so $\sigma_{h_H(G)}(e) = -1$. Thus $\sigma_G \equiv \sigma_{h_H(G)}$ so $G = h_H(G)$..

4 $\implies$ 1: Any all-positive graph is balanced (all walks are positive). After switching any vertex, all walks with exactly one endpoint at that vertex gets their sign flipped, since that vertex is incident to an odd number of edges in the walk. The other walks are unaffected since the switched vertex is incident to an even number of edges in the walk. Thus if the original graph were balanced, then the switched graph is also balanced. Thus all graphs switch equivalent to an all-positive graph are balanced. □

We will now extend vertex colouring to signed graphs.

2.2.1 Signed Colouring

Signed colouring was first introduced by Zaslavsky in 1982 [20] in the context of a signed extension of the chromatic polynomial. Other definitions of signed colouring include that introduced by Máčajová et al in 2014 [16] and by Naserasr et al in 2020 [18]. The main problem is that it is non-obvious how negative edges should interact with an odd number of colours. A common theme, however, is that a signed colouring needs to be consistent under vertex switching. In other words, the signed chromatic number should not change after vertex switching, and a colouring of the signed graph before the switch should induce a colouring of the switched graph. With these in mind, we give a definition of an *even* signed colouring. This is equivalent to previous definitions restricted to even numbers of colours.

Definition 2.2.5. Given a signed graph G, a *(proper even) signed (vertex) colouring* of G in $2k$ colours is a map

$$c : V(G) \to \{-k, -(k-1), \ldots, -1, 1, 2, \ldots, k\},$$

such that for all $uv \in E(G)$,

$$c(u) \neq \sigma(uv)c(v).$$

We use $\chi(G)$ to denote the *signed chromatic number* of G, the minimal $2k$ for which the above c exists, when G is a signed graph. Recall that $\chi(G)$ also denotes the unsigned chromatic number when G is an unsigned graph.

In other words, colours are signed, with a positive edge forbidding the same colour, while a negative edge forbids opposite colours. In fact, we can formally prove that this is the only way to define an even signed colouring consistent under switching, where positive edges forbid neighbours from having the same colour. Indeed, let G be an arbitrary signed

graph and c be a mapping from $V(G)$ to $\{-k, -(k-1), \ldots, -1, 1, \ldots, k\}$. Suppose that c satisfies that for every positive edge uv,

$$c(u) \neq c(v),$$

and further suppose that after any vertex $v \in V(G)$ is switched, the above holds for c' where $c'(u) = c'(u) \forall u \neq v$ but $c'(v) = -c(v)$. Then since, after the switch, the constraint becomes about the originally negative edges, any $v \in V(G)$ needs to also satisfy

$$-c(v) = c'(v) \neq c'(u) = c(u)$$

if $uv \in E(G)$ is negative, which is precisely our definition.

The following are some applied examples that motivate signed colouring.

- The vertices are tasks to be scheduled to two machines. Positive edges join tasks that cannot be scheduled on the same machine at the same time, while negative edges join tasks that cannot be scheduled on different machines at the same time. Positive colours represent time slots for one machine and negative colours represent those for the other.
- The vertices are players to be put into teams which play one round against each other in pairs. Positive edges join players who do not want to be in the same team, while negative edges join players who do not to play against each other. Opposite colours represent teams playing against each other.

It is clear that, given c a signed colouring of G, if $v \in V(G)$ is switched, then it suffices to colour v with $-c(v)$ instead to obtain a colouring of $h_v(G)$. This implies that for balanced signed graphs G whose underlying graph has an even chromatic number, $\chi(G) = \chi(\underline{G})$. We shall see that in general the best upper bound on the signed chromatic number is $\chi(G) \leq 2\chi(\underline{G})$. For the lower bound, note that an all negative graph G has $\chi(G) = 2$ (by colouring all vertices with 1). In other words, $\chi(\underline{G})$ cannot lower bound $\chi(G)$ and we need to instead use the trivial $\chi(G^+)$ (due to $G^+ \subseteq G$, as G^+ is the positive subgraph of G).

Proposition 2.2.6 (Máčajová et al 2014 [16]). *For a signed graph G,*

$$\chi(G^+) \leq \chi(G) \leq 2\chi(\underline{G^+}) \leq 2\chi(\underline{G}),$$

with all inequalities having infinitely many tight examples.

Note that $\chi(\underline{G^+})$ and $\chi(G^+)$ only differ by up to 1 due to us only defining even signed chromatic numbers.

Proof. Since $G^+ \subseteq G$, $\chi(G^+) \leq \chi(G)$ (as there are fewer restrictions to satisfy). Any all-positive graph achieves this bound. The $2\chi(\underline{G^+}) \leq 2\chi(\underline{G})$ bound is similar.

If c is a proper unsigned colouring of G^+, i.e. a mapping

$$c : V(G) \to \{1, \ldots, k\}, c(u) \neq c(w) \forall uw \in E(G^+),$$

then the signed colouring on G

$$c' : V(G) \to \{-k, \ldots, -1, 1, \ldots, k\}, v \mapsto c(v) \forall v \in V(G)$$

satisfies

$$c(u) \neq c(w) \forall uw \in E(G), \sigma(uw) = 1,$$

and clearly $c(u) \neq -c(w)$ for any $u, v \in V(G)$ since both colours are positive. Thus c is a proper signed colouring on G so $\chi(G) \leq 2\chi(\underline{G^+})$.

For a graph achieving $\chi(G) = 2\chi(\underline{G^+})$, fix $n \geq 1$ and consider $\underline{G} = K_{n,\ldots,n}$ a complete n-partite graph with each part having size n. Order the parts and the vertices in each. Label the i-th vertex in the j-th part $v_{i,j}$, with $1 \leq i, j \leq n$. Define G by assigning (for $j \neq j'$)

$$\sigma(v_{i,j}v_{i',j'}) = \begin{cases} -1, & i = i' > 1; \\ 1, & i \neq i' \text{ or } i = i' = 1. \end{cases}$$

In other words, $G[\{v_{1,1}, v_{1,2}, \ldots, v_{1,n}\}]$ is a positively labelled K_n, $G[\{v_{i,1}, v_{i,2}, \ldots, v_{i,n}\}]$ is a negatively labelled K_n for $i > 1$, and all $v_{i,j}, v_{i',j'}$ for $i \neq i', j \neq j'$ are joined positively (and there is no edge between $v_{i,j}, v_{i',j'}$ when $j = j'$). Clearly G^+ can be n-coloured as an unsigned graph, by colouring each part in a distinct colour. Thus

$$2\chi(\underline{G^+}) \leq 2n.$$

Suppose

$$c : V(G) \to \{-k, \ldots, -1, 1, \ldots, k\}$$

is a signed colouring of G, and for contradiction suppose $k < n$.

Let $c_i := c(v_{1,i})$. Then since $v_{1,1}, \ldots, v_{1,n}$ are all-positively adjacent, $c_1, \ldots, c_n$ are all distinct. For each $n \geq i > 1$, since less than n distinct absolute values are used by c, there exist two vertices in $G[\{v_{i,1}, \ldots, v_{i,n}\}]$ coloured using the same absolute value. But since

that subgraph is an all-negative K_n, those two vertices must have used the same colour, call it d_i.

Now for $i, i' > 1$, $i \neq i'$, if $d_i = d_{i'}$ then there exist $v_{i,j}, v_{i,j'}, v_{i',k}, v_{i',k'}$ all coloured d_i. This is impossible since $j \neq k$ or $j \neq k'$ must be true, so $v_{i,j}, v_{i',k}$ or $v_{i,j}, v_{i',k'}$ would be joined positively. Thus all d_i are distinct. Similarly, if $d_i = c_j$ then since there exist $v_{i,k}, v_{i,k'}$ both coloured d_i, while $v_{1,j}$ is coloured $c_j = d_i$, either $j \neq k$ or $j \neq k'$ must be true, so $v_{1,j}, v_{i,k}$ or $v_{1,j}, v_{i,k'}$ must be joined positively. Thus all d_i are distinct from $c_1, \ldots, c_n$. But then $\{c_1, \ldots, c_n, d_2, \ldots, d_n\} \subseteq \{-k, \ldots, -1, 1, \ldots, k\}$ which is impossible since the left side has $n + n - 1 = 2n - 1$ elements while the right has at most $2(n-1)$ elements.

Thus c uses at least $2n$ colours, i.e.

$$2n \leq \chi(G) \leq 2\chi(\underline{G^+}) \leq 2n$$

so $\chi(G) = 2\chi(\underline{G^+})$. □

Due to the very specific construction of the example, one may naturally suspect that it is rare that $\chi(G) = 2\chi(\underline{G^+})$, or that even $\chi(G) > \alpha\chi(\underline{G^+})$ for any $\alpha > 1$ might be rare. This motivates our investigation using random signed graphs. Another noteworthy fact is that the example has

$$|E(G)| = \frac{n^2 n(n-1)}{2} = (1 - o(1))\frac{n^4}{2}, |E(G^-)| = \frac{(n-1)n(n-1)}{2} = (1 - o(1))\frac{n^3}{2}$$

so $|E(G^-)| \sim \frac{|E(G)|}{n}$. This means that having few negative edges does not directly prevent a graph from having $\chi(G) = 2\chi(\underline{G^+})$. This means that our upcoming choice of $q = o(1)$ (where q is the probability that an edge in the random signed graph is labelled negative) is meaningful.

2.3 Probability Theory

The study of random graphs employs many advanced tools from probability theory. Here, we will be introducing conditional expectations and martingales along with standard results, which are crucial to our proof. We point to [5], [9], and [19] for reference and further reading. Recall that a probability space involves a *sample space* Ω, a *σ-algebra* $\mathcal{F}$ over Ω called the *event space*, and a *probability measure* $\mathbf{P} : \mathcal{F} \to [0, 1]$. In particular, in preparation for conditional expectations and martingales, the theory of σ-algebras requires special attention.

2.3.1 σ-Algebras and Measurability

Intuitively, a σ-algebra $\mathcal{F}$ over Ω is a collection of subsets of Ω, closed under certain operations as to allow the definition of a measure over $\mathcal{F}$. The pair $(\Omega, \mathcal{F})$ is then called a *measurable space*.

Definition 2.3.1. Given a set Ω, a *σ-algebra* $\mathcal{F}$ (over Ω) is a non-empty collection of subsets of Ω satisfying the following axioms:

1. If $E \in \mathcal{F}$ then $E^{\complement} \in \mathcal{F}$ (*closure under complement*).
2. If $E_j \in \mathcal{F}$ for all $j \geq 1$ then $\bigcup_{j=1}^{\infty} E_j \in \mathcal{F}$ (*closure under countable union*).

For a measure defined on $\mathcal{F}$, axiom 2 ensures that addition is well-defined, and axiom 1 ensures that subtraction is well-defined.

Note that axioms 1 and 2 imply that if $E_j \in \mathcal{F}$ for all $j \geq 1$, then $(\bigcup_{j=1}^{\infty} E_j^{\complement})^{\complement} = \bigcap_{j=1}^{\infty} E_j \in \mathcal{F}$. This is equivalent to axiom 2 (given axiom 1).

Since $\mathcal{F}$ is non-empty, there exists $E \in \mathcal{F}$, and by axiom 1 $E^{\complement} \in \mathcal{F}$, so by axiom 2 $E \cup E^{\complement} = \Omega \in \mathcal{F}$. By axiom 1 $\Omega^{\complement} = \varnothing \in \mathcal{F}$. Either of these is equivalent to the non-empty assumption.

To give some examples of σ-algebras, the smallest is $\mathcal{F} = \{\varnothing, \Omega\}$ and the largest is $\mathcal{F} = 2^{\Omega}$ the power set of Ω. The more interesting ones often sit somewhere in between. In particular, an important type of σ-algebra is the *smallest* one containing some given sets.

Definition 2.3.2. Given Ω and $\mathcal{C}$ a collection of subsets of Ω, the *σ-algebra generated by* $\mathcal{C}$, denoted $\sigma(\mathcal{C})$, is the intersection of all σ-algebras over Ω containing $\mathcal{C}$.

One can easily check that $\sigma(\mathcal{C})$ is a σ-algebra.

It is often hard to check for closure under countable union. Luckily, it turns out that we can break down that axiom into two weaker parts.

Definition 2.3.3. Given a set Ω, a *π-system* $\mathcal{I}$ (over Ω) is a collection of subsets of Ω *closed under finite intersection*, meaning, if $A, B \in \mathcal{I}$, then $A \cap B \in \mathcal{I}$.

It is clear that, by induction, closure under intersection of any two sets implies closure under intersection of any finite number of sets. Some definitions give non-emptiness as part of the definition of a π-system, but it will not be necessary for our purposes.

Definition 2.3.4. Given a set Ω, a *λ-system* $\mathcal{D}$ (over Ω) is a non-empty collection of subsets of Ω satisfying the following axioms:

1. If $E \in \mathcal{F}$ then $E^{\complement} \in \mathcal{F}$ (*closure under complement*).
2. If $E_j \in \mathcal{F}$ for all $j \geq 1$, and $E_j \cap E_i = \varnothing$ for all $j \neq i$, then $\bigsqcup_{j=1}^{\infty} E_j \in \mathcal{F}$ (*closure under disjoint countable union*).

Note that, similar to σ-algebras, since $\mathcal{F}$ is non-empty, $E, E^{\complement} \in \mathcal{F} \implies E \sqcup E^{\complement} = \Omega \in \mathcal{F}$, which is sometimes stated as an axiom instead of non-emptiness. Furthermore, given $A, B \in \mathcal{F}$, $B \subseteq A$, we have $A^{\complement} \in \mathcal{F}$ (by axiom 1) and $B \cap A^{\complement} = \varnothing$ so $B \cup A^{\complement} \in \mathcal{F}$ (by axiom 2), so $A \setminus B = (A^{\complement} \cup B)^{\complement} \in \mathcal{F}$ (by axiom 1) (*closure under complement of subsets*). When $\Omega \in \mathcal{F}$ is stated as an axiom, closure under complement of subsets is sometimes given instead of closure under complement.

One can clearly see the analogy between the axioms for a λ-system and those for a σ-algebra. Indeed, λ-systems only requiring closure under *disjoint* countable union is the only thing preventing λ-systems to be equivalent to σ-algebras, and the closure under finite intersection axiom of π-systems fixes exactly that.

Proposition 2.3.5. *Given a set Ω, a collection of subsets $\mathcal{F}$ is a σ-algebra if and only if it is both a π-system and a λ-system.*

Proof. First assume $\mathcal{F}$ is both a π-system and a λ-system. Then it is non-empty and we check the axioms:

1. If $E \in \mathcal{F}$ then $E^{\complement} \in \mathcal{F}$ (this axiom is the same).

2. Suppose $E_j \in \mathcal{F}$ for all $j \geq 1$. Then for every $j > 1$, $E_j \setminus (\bigcup_{i=1}^{j-1} E_i) = (\bigcup_{i=1}^{j} E_i) \setminus (\bigcup_{i=1}^{j-1} E_i) \in \mathcal{F}$ (by closure under complement of subsets).

   Call $A_1 := E_1$ and $A_j := E_j \setminus (\bigcup_{i=1}^{j-1} E_i)$ for $j > 1$. Then $A_j \in \mathcal{F}$ for all $j \geq 1$ and they are all disjoint ($A_i \subseteq E_i$ for all $i \geq 1$, and $A_j \cap E_i = \varnothing$ for all $j > i$). Thus by closure under disjoint countable union $\bigsqcup_{j=1}^{\infty} A_j \in \mathcal{F}$.

   Finally $\bigcup_{j=1}^{\infty} E_j = \bigsqcup_{j=1}^{\infty} A_j$ since clearly $\bigsqcup_{j=1}^{\infty} A_j \subseteq \bigcup_{j=1}^{\infty} E_j$, and if $x \in \bigcup_{j=1}^{\infty} E_j$ then let j_0 be the least j for which $x \in E_j$. Then $x \in A_j = E_j \setminus (\bigcup_{i=1}^{j-1} E_i)$ so $x \in \bigcup_{j=1}^{\infty} A_j$.

Thus $\mathcal{F}$ is a σ-algebra. The *only if* direction is clear. □

The usefulness of this proposition is clear: in order to generate a σ-algebra from a π-system, it suffices to generate a λ-system.

Theorem 2.3.6 (Dynkin's π-λ theorem)**.** *If $\mathcal{I}$ is a π-system and $\mathcal{I} \subseteq \mathcal{D}$ which is a λ-system, then $\sigma(\mathcal{I}) \subseteq \mathcal{D}$.*

Equivalently, since σ-algebras are themselves λ-systems, $\sigma(\mathcal{I})$ is precisely the smallest λ-system containing $\mathcal{I}$. Similar to $\sigma(\mathcal{I})$, it is easy to check that any intersection of λ-systems (over the same set) is a λ-system.

Proof. Let $\mathcal{D}_0$ be the smallest λ-system containing $\mathcal{I}$. If it is also a π-system then by Proposition 2.3.5 $\mathcal{D}_0$ is a σ-algebra so $\sigma(\mathcal{I}) = \mathcal{D}_0 \subseteq \mathcal{D}$ (by minimality of both $\sigma(\mathcal{I})$ and $\mathcal{D}_0$). Thus we check that $\mathcal{D}_0$ is a π-system.

For every $A \in \mathcal{D}_0$, define

$$\mathcal{D}_A := \{B \in \mathcal{D}_0 : A \cap B \in \mathcal{D}_0\}.$$

Since $A \in \mathcal{D}_A$, it is non-empty. We check that it is a λ-system:

1. If $E \in \mathcal{D}_A$ then $A \cap E \in \mathcal{D}_0$ and $A \cap E \subseteq A$, so $A \cap E^{\complement} = A \setminus (A \cap E) \in \mathcal{D}_0$ by closure under complement of subsets. So $E^{\complement} \in \mathcal{D}_A$ ($E^{\complement} \in \mathcal{D}_0$ is obvious).

2. If $E_j \in \mathcal{D}_A$ for all $j \geq 1$, and $E_j \cap E_i = \varnothing$ for all $j \neq i$, then $A \cap E_j \in \mathcal{D}_0$ for all $j \geq 1$ and those are disjoint as well, so $\bigsqcup_{j=1}^{\infty}(A \cap E_j) = A \cap (\bigsqcup_{j=1}^{\infty} E_j) \in \mathcal{D}_0$. So $\bigsqcup_{j=1}^{\infty} E_j \in \mathcal{D}_A$ ($\bigsqcup_{j=1}^{\infty} E_j \in \mathcal{D}_0$ is obvious).

Thus $\mathcal{D}_A$ is a λ-system. First assume $A \in \mathcal{I}$. Then by closure under finite intersection $A \cap B \in \mathcal{I} \subseteq \mathcal{D}_0$ for all $B \in \mathcal{I}$, so $\mathcal{I} \subseteq \mathcal{D}_A$. This means $\mathcal{D}_0 \subseteq \mathcal{D}_A$ by minimality of $\mathcal{D}_0$, but $\mathcal{D}_A \subseteq \mathcal{D}_0$ by definition, so $\mathcal{D}_A = \mathcal{D}_0$.

Now, take any $B \in \mathcal{D}_0 = \mathcal{D}_A$. Then $A \in \mathcal{D}_B$ by definition (since $A \cap B \in \mathcal{D}_0$) and that is true for any $A \in \mathcal{I}$, so $I \subseteq \mathcal{D}_B$. Again by minimality of $\mathcal{D}_0$, $\mathcal{D}_B = \mathcal{D}_0$. Thus, for any $B, C \in \mathcal{D}_0$, $C \in \mathcal{D}_B$ i.e. $B \cap C \in \mathcal{D}_0$ so it is a π-system. □

Another important concept related to measurable spaces $(\Omega, \mathcal{F})$ is functions between measurable spaces. In particular, random variables and Lebesgue-measurable functions are examples of those.

Given a pair of measurable spaces $(S, \mathcal{G}), (\Omega, \mathcal{F})$ and a function $X : \Omega \to S$, we want $\mathcal{F}$ to be "rich enough" to describe the behaviour of X on $\mathcal{G}$. More precisely, given any $B \in \mathcal{G}$, we want $X^{-1}(B)$, its preimage, to be measurable, i.e. to be in $\mathcal{F}$.

The idea is that we want $\mathcal{F}$ to give us enough information about X. Of course, if we know ω then we know $X(\omega)$. However, the idea is that by knowing whether ω is in A for every $A \in \mathcal{F}$, we also know $X(\omega)$ (or more precisely the membership of $X(\omega)$ in every set we care about, i.e. $\mathcal{G}$), without ever directly knowing ω. For a concrete example, picture a pseudo-random dice thrower which takes the amount of time passed since midnight scaled to $[0, 1)$, $\omega \in [0, 1)$, as input, and outputs $X(\omega)$, for some $X : [0, 1) \to \{1, \ldots, n\}$. It is clear that one can know what $X(\omega)$ is without knowing ω precisely, as long as one knows whether $\omega \in X^{-1}(\{k\})$ for every $k \in \{1, \ldots, n\}$. In this example, just $\sigma(X^{-1}(\{1\}), \ldots, X^{-1}(\{n\}))$ is sufficient to describe the behaviour of X.

Definition 2.3.7. Given $(\Omega, \mathcal{F})$ and $(S, \mathcal{G})$ measurable spaces, a function

$$X : \Omega \to S$$

is called *($\mathcal{F}$-)measurable (w.r.t. $\mathcal{G}$)* (written $X \in m\mathcal{F}$ and $X : (\Omega, \mathcal{F}) \to (S, \mathcal{G})$) if, for all $B \in \mathcal{G}$,

$$X^{-1}(B) \in \mathcal{F}.$$

If furthermore both Ω and S are topological spaces, and both $\mathcal{F} = \mathcal{B}(\Omega)$ and $\mathcal{G} = \mathcal{B}(S)$, then X is called a *Borel function*, where

$$\mathcal{B}(S) = \sigma(\{O \subseteq S : O \text{ open}\})$$

is the σ-algebra consisting of the *Borel sets* in S. In that case, $(S, \mathcal{G})$ is called a *Borel space*.

When $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space and $(S, \mathcal{G})$ is a Borel space, X is called an *(S-valued) random variable*.

Intuitively, a real-valued random variable X is simply a random number in $\mathbb{R}$. The measurability requirements ensure that we have a probability space sufficiently equipped to talk about the probabilistic properties of X. Else, we might not have a $\mathbf{P}\{X \in B\}$ for every Borel set B. We care about *Borel sets* in particular, because, for many spaces S (including $\mathbb{R}$), $\mathcal{B}(S)$ encompass virtually all the interesting sets, despite $\mathcal{B}(S)$ being the minimal choice if one wants to at least include open sets.

Furthermore, it is not difficult to show that continuous functions are Borel, and that the composition of $f : (S_1, \mathcal{F}_1) \to (S_2, \mathcal{F}_2)$ with $g : (S_2, \mathcal{F}_2) \to (S_3, \mathcal{F}_3)$ is $g \circ f : (S_1, \mathcal{F}_1) \to (S_3, \mathcal{F}_3)$. This shows that Borel functions of random variables are still random variables. Intuitively, if we have a random outcome, then any function applied to that random outcome is still just another random outcome. If the function is nice enough (Borel), then the same collection of events which was sufficient to have complete knowledge of the previous random outcome also gives complete knowledge of the new random outcome. For an example as to why the function applied to the random variable needs to be Borel, let $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be uniformly random in $[0, 1]$, and let $A \subseteq [0, 1]$ be a non-Borel set. Then X is a random variable, but not necessarily $\mathbf{1}_A(X)$, as $\mathbf{1}_A(X)^{-1}(\{1\}) = X^{-1}(A)$ is not known to be in $\mathcal{F}$, since $A \notin \mathcal{B}(\mathbb{R})$.

In general, given a function $X : \Omega \to S$, with $(S, \mathcal{G})$ being a measurable space, one might be interested in the *smallest* σ-algebra needed on Ω so that X is measurable. In other words, that σ-algebra consists of all subsets of Ω "discernable" by looking at the outcome of X in terms of membership in $\mathcal{G}$.

Definition 2.3.8. Given $X : \Omega \to S$, with $(S, \mathcal{G})$ measurable space, define the *σ-algebra generated by X (under $\mathcal{G}$)*

$$\sigma(X) := \sigma(\{X^{-1}(B) : B \in \mathcal{G}\}).$$

Also note that when σ takes multiple arguments, e.g. $\sigma(X, \mathcal{C})$, we mean $\sigma(\sigma(X) \cup \sigma(\mathcal{C}))$.

With these definitions and results, we will now move on to conditional expectations and martingales, which rely heavily on measurability.

2.3.2 Conditional Expectation

Conditional expectations, intuitively, are simply the "best guess" possible for the value of a random variable, given the information known. For example, that guess, given *zero*

information, is just the usual expectation. For a non-trivial example, if we try to guess the result of a dice throw given that it is even, we would be guessing 4, the expectations of even throws, instead of 3.5, the overall expectation.

An intuitive way to define conditional expectation could be as follows. Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, a random variable X, and an event $A \in \mathcal{F}$ with non-zero probability, the expectation of X conditioning on A is

$$\frac{\mathbf{E}_A\{X\}}{\mathbf{P}\{A\}}.$$

Indeed, $\mathbf{E}_A\{X\}$ integrates X over ω for which A happens, and $\mathbf{P}\{A\}$ is the area of such ω, so the above *is* the average value of X when A happens. For the dice throw example, X is the outcome of the dice throw, $A = \{\text{even}\}$, so

$$\mathbf{E}_A\{X\} = \frac{1}{6}(2+4+6) = \frac{4}{2}$$

while $\mathbf{P}\{A\} = \frac{1}{2}$.

A big drawback, however, is that this definition only works with conditioning on the occurrence or not of *one* event. What if we would like to condition on the outcome of multiple events (for example, even and ≥ 3), or even on the value of another random variable? The result should be a random variable, completely determined by the outcome of these events or by the value of that random variable that we are conditioning on. This motivates the following definition.

Definition 2.3.9. Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, and a random variable X with finite expectation, the *conditional expectation of X w.r.t. $\mathcal{G}$* is any random variable $Y \in m\mathcal{G}$, satisfying

$$\mathbf{E}_A\{Y\} = \mathbf{E}_A\{X\}$$

for all $A \in \mathcal{G}$. We denote $Y = \mathbf{E}\{X \mid \mathcal{G}\}$. Furthermore, for Z another random variable, we use the shorthand

$$\mathbf{E}\{X \mid \sigma(Z)\} = \mathbf{E}\{X \mid Z\}.$$

Let us break down this definition. As an aid, let us consider the example of X being the outcome of a dice throw, and $\mathcal{G} = \sigma(\{\text{even}\}, \{\geq 3\})$. The random variable Y satisfies two things.

First, it needs to be $\mathcal{G}$-measurable. In other words, knowledge of $\mathcal{G}$ (the occurrence or not of every event in $\mathcal{G}$) is sufficient to determine the value of Y. In the example, $\mathcal{G} = \sigma(\{\text{even}, \{\geq 3\}\})$, so knowing whether the dice throw was even and whether it was no less than 3 is sufficient to know Y:

$$\begin{aligned} \{\text{even}\} \cap \{X \geq 3\} &\implies Y = 5; \\ \{\text{odd}\} \cap \{X \geq 3\} &\implies Y = 4; \\ \{\text{even}\} \cap \{X < 3\} &\implies Y = 2; \\ \{\text{odd}\} \cap \{X < 3\} &\implies Y = 1. \end{aligned}$$

Second, it needs to agree with X in expectation on $\mathcal{G}$. What this means is that for any event in $\mathcal{G}$, the intuitive definition of conditional expectation applies. When the outcomes of everything in $\mathcal{G}$ become known, the value of Y also becomes known and it is equal to the expectation of X conditioning (with the intuitive definition) on those outcomes, i.e. the average value of X on those outcomes, as the example above shows.

For our purposes, since Ω will be a finite set, the existence and uniqueness of a conditional expectation are obvious (since the value of Y is determined for every atomic event in $\mathcal{G}$). The reference cited [19] contains a general proof of those facts.

Analogous to the usual expectation, conditional expectations are also linear.

Proposition 2.3.10 (Linearity of conditional expectation)**.** *Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, random variables X, Y with finite expectation, and $a \in \mathbb{R}$,*

$$a\mathbf{E}\{X \mid \mathcal{G}\} + \mathbf{E}\{Y \mid \mathcal{G}\} = \mathbf{E}\{aX + Y \mid \mathcal{G}\}.$$

Proof. We remark that $a\mathbf{E}\{X \mid \mathcal{G}\} + \mathbf{E}\{Y \mid \mathcal{G}\}$ meets the definition. Indeed it is $\mathcal{G}$-measurable (as a Borel function of $\mathcal{G}$-measurable functions) and agrees with $aX + Y$ on any $A \in \mathcal{G}$ by linearity of expectation. □

A property unique to conditional expectations is the *tower property*. Intuitively, it means that when we repeatedly do conditional expectations on nested σ-algebras, the smallest (coarsest, the one with the least information) one dominates.

Proposition 2.3.11 (Tower property)**.** *Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, a random variable X with finite expectation, and σ-algebras $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$, we have*

$$\mathbf{E}\{\mathbf{E}\{X \mid \mathcal{G}\} \mid \mathcal{H}\} = \mathbf{E}\{X \mid \mathcal{H}\} = \mathbf{E}\{\mathbf{E}\{X \mid \mathcal{H}\} \mid \mathcal{G}\}.$$

In particular, $\mathbf{E}\{\mathbf{E}\{X \mid \mathcal{G}\}\} = \mathbf{E}\{X\}$.

Proof. Second equality: Since $\mathbf{E}\{X \mid \mathcal{H}\}$ is already $\mathcal{G}$-measurable (for all B Borel, $\mathbf{E}\{X \mid \mathcal{H}\}^{-1}(B) \in \mathcal{H} \subseteq \mathcal{G}$), its conditional expectation w.r.t. $\mathcal{G}$ is itself.

First equality: $\mathbf{E}\{X \mid \mathcal{H}\}$ is $\mathcal{H}$-measurable. For all $A \in \mathcal{H}$, $\mathbf{E}_A\{\mathbf{E}\{X \mid \mathcal{H}\}\} = \mathbf{E}_A\{X\}$ by definition, and $\mathbf{E}_A\{X\} = \mathbf{E}_A\{\mathbf{E}\{X \mid \mathcal{G}\}\}$ by definition as well since $A \in \mathcal{H} \subseteq \mathcal{G}$, so

$$\mathbf{E}_A\{\mathbf{E}\{X \mid \mathcal{H}\}\} = \mathbf{E}_A\{\mathbf{E}\{X \mid \mathcal{G}\}\}.$$

This shows $\mathbf{E}\{X \mid \mathcal{H}\}$ meets the definition for $\mathbf{E}\{\mathbf{E}\{X \mid \mathcal{G}\} \mid \mathcal{H}\}$. □

A second important property is the fact that conditioning on a σ-algebra that is independent to both the random variable and what is currently known will bring no new "information". In other words, if $\mathcal{G}$ is independent from X and $\mathcal{H}$ simultaneously, conditioning on $\sigma(\mathcal{G}, \mathcal{H})$ is no different from conditioning on just $\mathcal{H}$.

Proposition 2.3.12. *Let $\mathcal{H}, \mathcal{G}$ be σ-algebras and X be a random variable, such that $\mathcal{G}$ is independent from $\sigma(X, \mathcal{H})$. Then*

$$\mathbf{E}\{X \mid \sigma(\mathcal{H}, \mathcal{G})\} = \mathbf{E}\{X \mid \mathcal{H}\}.$$

In particular, $\mathbf{E}\{X \mid \mathcal{G}\} = \mathbf{E}\{X\}$.

Proof. Since $\mathbf{E}\{X \mid \mathcal{H}\} \in m\mathcal{H} \subseteq m\sigma(\mathcal{H}, \mathcal{G})$, it suffices to prove that for all $E \in \sigma(\mathcal{H}, \mathcal{G})$,

$$\mathbf{E}_E\{X\} = \mathbf{E}_E\{\mathbf{E}\{X \mid \mathcal{H}\}\}. \tag{2.3.1}$$

Call $\mathcal{D}$ the collection of sets where this equality holds. Define $\mathcal{I} := \{A \cap B : A \in \mathcal{H}, B \in \mathcal{G}\}$. Note $\mathcal{I}$ is a π-system (if $A_1, A_2 \in \mathcal{H}$ and $B_1, B_2 \in \mathcal{G}$, then $A_1 \cap A_2 \in \mathcal{H}, B_1 \cap B_2 \in \mathcal{G}$ and $(A_1 \cap B_1) \cap (A_2 \cap B_2) = (A_1 \cap A_2) \cap (B_1 \cap B_2) \in \mathcal{I}$). Also, for all $A \in \mathcal{H}, B \in \mathcal{G}$,

$$\mathbf{E}_{A\cap B}\{X\} = \mathbf{P}\{B\}\mathbf{E}_A\{X\} = \mathbf{P}\{B\}\mathbf{E}_A\{\mathbf{E}\{X \mid \mathcal{H}\}\} = \mathbf{E}_{A\cap B}\{\mathbf{E}\{X \mid \mathcal{H}\}\}.$$

The middle equality is because $A \in \mathcal{H}$; the first and third equalities are because B is independent from $\sigma(X, \mathcal{H})$ while $X\mathbf{1}_A, \mathbf{E}\{X \mid \mathcal{H}\}\mathbf{1}_A \in m\sigma(X, \mathcal{H})$, so they are independent from $\mathbf{1}_B$. Thus $\mathcal{I} \subseteq \mathcal{C}$. We now check $\mathcal{D}$ is a λ-system:

1. If $E \in \mathcal{D}$, then

   $$\mathbf{E}_{E^{\complement}}\{X\} = \mathbf{E}\{X\} - \mathbf{E}_E\{X\} = \mathbf{E}\{\mathbf{E}\{X \mid \mathcal{H}\}\} - \mathbf{E}_E\{\mathbf{E}\{X \mid \mathcal{H}\}\} = \mathbf{E}_{E^{\complement}}\{\mathbf{E}\{X \mid \mathcal{H}\}\}$$

   so $E^{\complement} \in \mathcal{D}$.

2. If $E_j \in \mathcal{C}$ for all $j \geq 1$ and $E_j \cap E_i = \varnothing$ then note that by linearity and definition of $\mathcal{C}$

   $$\mathbf{E}\{X\mathbf{1}_{\bigcup_{i=1}^{j} E_i}\} = \sum_{i=1}^{j} \mathbf{E}\{X\mathbf{1}_{E_i}\} = \sum_{i=1}^{j} \mathbf{E}\{\mathbf{E}\{X \mid \mathcal{H}\}\mathbf{1}_{E_i}\} = \mathbf{E}\{\mathbf{E}\{X \mid \mathcal{H}\}\mathbf{1}_{\bigcup_{i=1}^{j} E_i}\}$$

   for all $j \geq 1$. Furthermore $X\mathbf{1}_{\bigcup_{i=1}^{j} E_i}$ and $X \mid \mathcal{H}\}\mathbf{1}_{\bigcup_{i=1}^{j} E_i}$ are both dominated (one by $|X|$ and the other by $|\mathbf{E}\{X \mid \mathcal{H}\}|$). Thus by dominated convergence

   $$\begin{aligned}\mathbf{E}\{X\mathbf{1}_{\bigcup_{j=1}^{\infty} E_j}\} &= \lim_{j\to\infty} \mathbf{E}\{X\mathbf{1}_{\bigcup_{i=1}^{j} E_i}\} \\ &= \lim_{j\to\infty} \mathbf{E}\{\mathbf{E}\{X \mid \mathcal{H}\}\mathbf{1}_{\bigcup_{i=1}^{j} E_i}\} = \mathbf{E}\{\mathbf{E}\{X \mid \mathcal{H}\}\mathbf{1}_{\bigcup_{j=1}^{\infty} E_j}\}\end{aligned}$$

   so $\bigcup_{j=1}^{\infty} E_j \in \mathcal{D}$.

Thus $\mathcal{D}$ is a λ-system containing $\mathcal{I}$. By Dynkin's π-λ theorem Theorem 2.3.6 $\sigma(\mathcal{I}) \subseteq \mathcal{D}$. Since $\mathcal{H}, \mathcal{G} \subseteq \mathcal{I}$, $\sigma(\mathcal{H}, \mathcal{G}) \subseteq \sigma(\mathcal{I}) \subseteq \mathcal{D}$ so equality (2.3.1) holds on $\sigma(\mathcal{H}, \mathcal{G})$. □

$\mathcal{G}$ being independent from $\sigma(X, \mathcal{H})$ is necessary, and neither $\mathcal{G}$ being independent from $X, \mathcal{H}$ nor even the three being pairwise independent is strong enough. Indeed a counterexample can be constructed from any three pairwise but not mutually independent events. Consider for example the probability space of two independent fair coin throws. Let $A = \{\texttt{HH}, \texttt{TT}\}, B = \{\text{first is } \texttt{H}\}, C = \{\text{last is } \texttt{T}\}$. A, B, A, C and B, C are all pairwise independent. However, let $X = \mathbf{1}_A$, $\mathcal{H} = \sigma(B)$ and $\mathcal{G} = \sigma(C)$. Clearly $\mathcal{G}, \mathcal{H}, X$ are pairwise independent, but $\sigma(\mathcal{G}, \mathcal{H})$ contains full information about X.

2.3.3 Martingales

We are now almost ready to introduce martingales. A martingale, intuitively, is a sequence of random variables, indexed by time (so it is a *random process*), such that the expected value at any future point, conditioning on all the information available at present, is equal to the value at present. Examples include:

- The amount of money a gambler has while playing a fair game.
- The x-position of a particle in Brownian motion.

The sequence consisting of the sets of "information" at every point in time is formalized by a *filtration*.

Definition 2.3.13. Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, a *filtration* $\{\mathcal{F}_k\}_{k\geq 0}$ is an increasing sequence of sub-σ-algebras of $\mathcal{F}$, such that $\mathcal{F}_0 = \{\varnothing, \Omega\}$.

As we can see, at time 0 no information is available, and the sequence is increasing, i.e. one gains information as time advances. Note that $\mathcal{F}_0 = \{\varnothing, \Omega\}$ is purely by convention, as one can just add this $\mathcal{F}_0$ in front of any increasing sequence of σ-algebras.

Recall that a random variable X is *measurable* w.r.t. a σ-algebra $\mathcal{F}$ when its value (more precisely, the membership of $X(\omega)$ in every Borel set) is completely determined by the membership of ω in every $E \in \mathcal{F}$. Thus, $X \in m\mathcal{F}_t$ if and only if the value of X is completely determined at time t. Similarly, $\mathbf{E}\{X \mid \mathcal{F}_t\}$ is simply the "best guess" for the value of X, given the information available at time t.

A martingale w.r.t. a filtration, then, is a sequence of random variables such that the "best guess" for its value at a future time, is simply its value at the present time. In other words, if the current time is t (i.e. we can condition on $\mathcal{F}_t$), and $s \geq t$ (i.e. X_s is a future value), then the "best guess" for X_s at time t (which is by definition $\mathbf{E}\{X_s \mid \mathcal{F}_t\}$), is simply X_t.

Definition 2.3.14. Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\{\mathcal{F}_k\}_{k\geq 0}$, a *martingale* (w.r.t. filtration $\{\mathcal{F}_k\}_{k\geq 0}$) is a sequence of random variables $\{X_k\}_{k\geq 0}$, such that

$$\mathbf{E}\{X_j \mid \mathcal{F}_k\} = X_k$$

for all $j \geq k \geq 0$.

Back to the two examples above,

- X_t is the amount of money after playing t games, and the events $\mathcal{F}_t$ are the outcomes of the first t games.
- X_t is the position of the Brownian particle at time t, and $\mathcal{F}_t = \sigma(X_1, \ldots, X_t)$ is simply the path of the particle up to time t. Note that $\{X_t\}$ being a martingale is essentially saying, for $s \geq t$,
$$\mathbf{E}\{X_s \mid X_1, \ldots, X_t\} = X_t.$$
In other words, the best guess for the future x-position of a Brownian particle is its current position.

Notice that the $\mathcal{F}_0$ condition forces X_0 to be a.s. constant. Furthermore,

$$\mathbf{E}\{X_t\} = \mathbf{E}\{\mathbf{E}\{X_t \mid \mathcal{F}_0\}\} = \mathbf{E}\{X_0\}$$

for any $t \geq 0$ by definition. Thus $X_0 = \mathbf{E}\{X_t\}$ a.s. for every $t \geq 0$, i.e. X_0 is the expectation of the martingale.

Imagine a martingale where the difference between consecutive terms is bounded. Because a martingale is not expected to increase or decrease, its probability to increase or decrease by any $\epsilon > 0$ is also bounded: If $|X_{k+1} - X_k| \leq c$ then $\mathbf{P}\{X_{k+1} \geq X_k - \epsilon \mid \mathcal{F}_k\}, \mathbf{P}\{X_{k+1} \leq X_k + \epsilon \mid \mathcal{F}_k\}$ cannot be less than $\frac{\epsilon}{c+\epsilon}$. Indeed, if $X_{k+1} \geq X_k - \epsilon$ then $X_{k+1} - X_k \leq c$, and if $X_{k+1} < X_k - \epsilon$ then $X_{k+1} - X_k < -\epsilon$, so

$$0 = \mathbf{E}\{X_{k+1} - X_k \mid \mathcal{F}_k\} < c\mathbf{P}\{X_{k+1} \geq X_k - \epsilon \mid \mathcal{F}_k\} - \epsilon(1 - \mathbf{P}\{X_{k+1} \geq X_k - \epsilon \mid \mathcal{F}_k\})$$

which rearranges to

$$\mathbf{P}\{X_{k+1} \geq X_k - \epsilon \mid \mathcal{F}_k\} > \frac{\epsilon}{c+\epsilon}.$$

This means

$$\mathbf{P}\{X_{k+1} < X_k - \epsilon \mid \mathcal{F}_k\}, \mathbf{P}\{X_{k+1} > X_k + \epsilon \mid \mathcal{F}_k\} \leq \frac{c}{c+\epsilon}.$$

Therefore, the probability for the martingale to have wandered far above or below its expectation, i.e. the probability that it increased much more often than it had decreased or vice versa, should shrink very fast. This is the intuition behind the *Azuma-Hoeffding inequality*.

Theorem 2.3.15 (Azuma-Hoeffding inequality [4] [14]). *Suppose* $\{X_k\}_{k\geq 0}$ *is a martingale and* $|X_k - X_{k-1}| \leq c_k$ *a.s. for all* $k \geq 1$. *Then for all* $N \geq 1$ *and all* $a > 0$,

$$\mathbf{P}\{X_k - X_0 \geq a\} \leq \exp\left\{\frac{-a^2}{2\sum_{k=1}^{N} c_k^2}\right\},$$

$$\mathbf{P}\{X_0 - X_k \geq a\} \leq \exp\left\{\frac{-a^2}{2\sum_{k=1}^{N} c_k^2}\right\}.$$

Chapter 3

Model of Random Signed Graphs

Recall that the Erdős-Rényi model of random graphs, $G(n,p)$, consists of n vertices, with every pair of vertices joined by an edge independently with a probability p. A natural extension to signed graphs would be to simply take $G(n,p)$, and randomly label the edges positive or negative independently.

Definition 3.0.1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space consisting of $\binom{n}{2}$ $\mathbf{Ber}(p)$ random variables $\{X_e\}_{e\in E(K_n)}$, and another $\binom{n}{2}$ $\mathbf{Ber}(q)$ random variables $\{Y_e\}_{e\in E(K_n)}$, all independent. Then $G(n,p,q)$, mapping from $\Omega = \{0,1\}^{E(K_n)}$ to the set of graphs on n vertices, is such that for $e \in E(K_n)$,

$$\begin{cases} e \in E(G(n,p)), \sigma(e) = -1, & X_e = 1, Y_e = 1; \\ e \in E(G(n,p)), \sigma(e) = 1, & X_e = 1, Y_e = 0; \\ e \notin E(G(n,p)), & \text{else.} \end{cases}$$

In other words, we take $G(n,p)$ and flip a coin independently for every edge to determine its sign, with a probability q for it to be negative. This is slightly different from El Maftouhi's [17] model, in that they define $G'(n,p',q')$, with

$$p' = \mathbf{P}\{uv \in E(G), \sigma(uv) = 1\}, q' = \mathbf{P}\{uv \in E(G), \sigma(uv) = -1\}.$$

Clearly, their model would correspond to $G(n, p'+q', \frac{q'}{p'+q'})$ in our definition, and ours to $G'(n, p(1-q), pq)$ in theirs. All of our results can be restated under their model $G'(n,p,q)$

by replacing every p with $p+q$ and every q with $\frac{q}{p+q}$. We are using a modified model because we want the edge probability to be one parameter, so that $b = \frac{1}{1-p}$ only depends on p, as we will see in the next section.

The study of the unsigned chromatic number of $G(n,p)$ (which we undertake in detail in Chapter 4) is intrinsically linked to its independence number. This is because, for unsigned graph colourings, each colour class is an independent set. For signed colourings, it would not suffice to only look at individual colour classes since opposite colours interact with each other. However, it would be very natural to look at a signed colour class consisting of all vertices coloured with the same *absolute value*. The kind of sets that signed colour classes can take is precisely what we will call *signed independent sets*.

3.1 A Signed Analogue to Colour Classes

Definition 3.1.1. Given a signed graph G, $S \subseteq V(G)$ is a *signed independent set* if there exists $H \subseteq S$ such that $H, S \setminus H$ are independent sets in G^+, and there are no negative edges between H and $S \setminus H$.

One can easily check that by colouring H with colour i and $S \setminus H$ with $-i$, signed independent sets are precisely the kind of sets that can take the same absolute value in a signed colouring. However, an issue arises when we want to talk about signed independent sets of some fixed size. If we only talk about the size of S, then $H \subseteq S$ can be of any size, and the rarity of such S in $G(n,p,q)$ will largely depend on the size of H. Taking inspiration from the unsigned case, where we consider the number of independent sets of size r, we want both H and $S \setminus H$ to have size r, as they are both independent sets in G^+. This motivates the actual object of interest in our investigation of $G(n,p,q)$.

Definition 3.1.2. Given a signed graph G, the pair $\{H, S \setminus H\}$ is a *special independent pair* (of size $2r$) if S is a signed independent set of size $2r$, and $H \subseteq S$ has size r.

Note that the same signed independent set S could potentially contain multiple special independent pairs. Furthermore, for convenience, when $V(G)$ is ordered, we use H for the set in the pair with the smaller least vertex and use $S \setminus H$ for the one with the larger least vertex. Analogous to the unsigned case, we start our investigation of the signed

chromatic number of a random signed graph by computing the expected number of special independent pairs close to maximal size.

Note that a special independent pair of size $2r$ in a signed graph G contains in particular two independent sets of size r in G^+, so this maximal size is no more than twice the (unsigned) independence number of G^+. We can prove that we *do* expect many special independent pairs near this size.

Proposition 3.1.3. *Let $0 < p = p(n) < 1$ be asymptotically bounded away from $0, 1$. Let $b = \frac{1}{1-p}$ and $q = q(n) = o(1)$. Suppose that $c = c(n) > 0$ is such that there exists constant $C \in \mathbb{R}$, such that $\liminf 2(1 - 2\frac{\log c}{\log b}) \geq C$. Let*

$$r = r(n) = 2\log_b(ecn/(2\log_b n)).$$

Let X denote the number of special independent pairs of size $2r$ in $G(n, p, q)$. Then, for each $\epsilon > 0$ constant,

$$E(n, r) := \mathbf{E}X = \omega(n^{C-\epsilon}).$$

Proof. To get a special independent pair of size $2r$, there are $\binom{n}{2r}$ choices for S and $\frac{1}{2}\binom{2r}{r}$ choices for $H \subseteq S$ (since H always has the smaller least vertex among H and $S \setminus H$). Given a fixed choice of $H, S \setminus H$, for them to be a special independent pair, r^2 negative edges need to be missing between H and $S \setminus H$ and $\binom{r}{2}$ positive edges need to be missing inside H and $S \setminus H$. Thus

$$\begin{aligned}
E(n, r) &= \frac{1}{2}\binom{n}{2r}\binom{2r}{r}(1 - p(1-q))^{2\binom{r}{2}}(1-pq)^{r^2} \\
&= \frac{n!}{2(n-2r)!(r!)^2}(1 - p(1-q))^{r^2-r}(1-pq)^{r^2} \\
&= \frac{n!}{2(n-2r)!(r!)^2}(1 - p + pq)^{-r}((1-p+pq)(1-pq))^{r^2} \\
&= \frac{n^{2r}(1-o(1))}{2(r!)^2}(1-p+pq)^{-r}(1-p+p^2q(1-q))^{r^2}
\end{aligned}$$

and, similarly to the unsigned case,

$$(1-p+p^2q(1-q))^{r^2} \geq (1-p)^{r^2} = \exp\{r^2\log(1-p)\} = \left(\frac{ecn}{2\log_b n}\right)^{-2r}.$$

Note

$$(1-p+pq)^{-r} = \exp\left\{\log(1-p+pq)\frac{2\log(ecn/(2\log_b n))}{\log(1-p)}\right\} = \left(\frac{ecn}{2\log_b n}\right)^{2\log(1-p+pq)/\log(1-p)}$$

and clearly $\log(1-p+pq)/\log(1-p) = 1-o(1)$. So

$$\begin{aligned} E(n,r) &\geq \frac{n^{2r-2r}((2/ec)\log_b n)^{2r}(1-o(1))}{2(r!)^2}(1-p+pq)^{-r} \\ &= \frac{((2/ec)\log_b n)^{2r}}{2(r!)^2}\left(\frac{ecn}{2\log_b n}\right)^{2(1-o(1))} = \frac{((2/ec)\log_b n)^{2r}}{2(r!)^2}\omega(n^{2-\epsilon}). \end{aligned}$$

Note $r! \sim (r/e)^r\sqrt{2\pi r} = ((2/e)\log_b(ecn/(2\log_b n)))^r\sqrt{2\pi r}$, so

$$\begin{aligned} \frac{((2/ec)\log_b n)^{2r}}{(r!)^2} &= (1-o(1))\left(\frac{\log_b n}{c\log_b(ecn/(2\log_b n))}\right)^{2r}/(2\pi r) \\ &= \Omega((\log n)^{-1})c^{-2r} \\ &= \Omega((\log n)^{-1})\exp\{(-2\log c/\log b)(1-o(1))2\log n\} \\ &= \Omega((\log n)^{-1})n^{-\frac{4\log c}{\log b}(1-o(1))} \end{aligned}$$

Thus $E(n,r) = \omega(n^{C-\epsilon})$. □

Note that the b used in here depends on the edge probability, not just the positive edge probability. This justifies our choice of modifying $G(n,p,q)$ from [17] so that p is the edge probability and not the positive edge probability.

One cannot help but notice that Proposition 3.1.3 is almost exactly analogous to the lower bound part of Proposition 2.1.11, from the derivation to the result itself. Indeed, this asymptotic lower bound on $E(n,r)$ is at worst a subpolynomial away from being the square of the lower bound on $E'(n,r)$ from Proposition 2.1.11 applied to $G(n,p) \sim \underline{G(n,p,q)}$. For a fixed signed graph G, if G^+ has ℓ independent sets of size r, then G has at most $\binom{\ell}{2} = \Theta(\ell^2)$ special independent pairs of size $2r$ and this can only be achieved if those independent sets are disjoint, and there are *no* negative edges between any pair of those independent sets. If one wanted to apply Proposition 2.1.11 to $G^+ \sim G(n,p(1-q))$, the positive subgraph of $G(n,p,q)$, then b would be slightly smaller compared to $G(n,p)$, but

$$\liminf\left(1-\frac{2\log c}{\log\frac{1}{1-p(1-q)}}\right) = \liminf\left(1-\frac{2\log c}{\log\frac{1}{1-p}}\right)$$

given that p is asymptotically away from $0, 1$. In that case, the lower bound on the expected number of independent sets of size r in G^+ given by Proposition 2.1.11 is no different from those in $G(n,p,q)$, so the bound on $E(n,r)$ from $G(n,p,q)$ is also almost the square of $E'(n,r)$ from G^+. The fact that they almost achieve the maximal order of magnitude suggests that the negative edges have a small impact when $q = o(1)$, which hints at $\chi(G(n,p,q)) \sim \chi(G^+)$, and that the independent sets in $G(n,p)$ tend to be mostly disjoint, which is the foundation of Bollobás's analysis in Section 4.1.

3.2 Signed Chromatic Number Unchanged for Small q

Although it is true that the order of magnitude of the signed chromatic number will not change by labelling the negative edges with $q = o(1)$ (as we shall show later), it does not mean that it will be the same as the unsigned chromatic number. Nonetheless, we can prove separately that for small enough q, a.a.s. the signed chromatic number of $G(n,p,q)$ is equal (up to adding one new colour due to the unsigned chromatic number being odd) to the unsigned chromatic number of its positive subgraph G^+.

Intuitively, given an unsigned colouring c of G^+, its colour classes would also work on a signed colouring of $G(n,p,q)$ if and only if there exists a way to match them in pairs, such that there are no negative edges between the colour classes of the same pair. These pairs would become the signed independent sets. This sounds strikingly like a perfect matching in some auxiliary graph.

Indeed, we can consider an auxiliary graph where the vertices are the colour classes of c, and two colour classes are adjacent if and only if there are no negative edges between the colour classes in $G(n,p,q)$. Given n, p, q, although this auxiliary graph is not exactly $G(n',p')$ (its number of vertices is not constant and the edge probability also depends on the endpoints), we can find n', p' such that a.a.s. the subgraph of $G(n',p')$ induced by the vertex set of our auxiliary graph can be embedded as a subgraph of our auxiliary graph, such that $G(n',p')$ having a perfect matching implies that our auxiliary graph also has a perfect matching.

Proposition 3.2.1. *Let $0 < p = p(n) < 1$ be asymptotically bounded away from $0, 1$ and $b = \frac{1}{1-p}$. Suppose there exists $C \in \mathbb{R}$ constant such that a.a.s.*

$$\chi(G(n, p(1-q))) \leq \frac{n}{2\log_b n - 2\log_b \log_b n + C}$$

(this C exists by Theorem 4.1.2). If

$$q \leq \frac{1}{4p \log_b n}$$

for all large enough n, then a.a.s. $\chi(G^+) = \chi(G)$ (i.e. $\chi(\underline{G^+}) \leq \chi(G) \leq \chi(\underline{G^+}) + 1$), where $G \sim G(n,p,q)$, and G^+ denotes its positive subgraph.

Proof. Let c be an optimal unsigned colouring of G^+. Then recall from Proposition 2.1.11 a.a.s. each colour class of c has size no more than

$$r(n) = 2\log_b n - 2\log_b \log_b n + C'$$

(the upper bound on the independence number of G^+) for some constant $C' \geq C$ depending only on p. Define an auxiliary (unsigned) graph Γ_n on $\chi(\underline{G^+})$ vertices, such that vertices i, j are adjacent if and only if there are no negative edges between the i-coloured and j-coloured colour classes. Then, since c uses no more than $\frac{n}{2\log_b n - 2\log_b \log_b n + C}$ colours, and (let S_i, S_j denote the colour classes for i, j) a.a.s.

$$\mathbf{P}\{i \sim j \text{ in } \Gamma_n\} = (1-pq)^{|S_i||S_j|} \geq (1-pq)^{(2\log_b n - 2\log_b \log_b n + C')^2} =: p'.$$

Thus there exists a coupling of Γ_n and $G(\chi(\underline{G^+}), p')$, such that a.a.s. Γ_n is a supergraph of $G(\chi(\underline{G^+}), p')$ on the same vertex set, so there exists a further coupling with $G(n', p')$ ($n' = \lfloor \frac{n}{2\log_b n - 2\log_b \log_b n + C} \rfloor$), so that a.a.s. Γ_n is a supergraph of the subgraph induced by the first $\chi(\underline{G^+})$ vertices of $G(n', p')$. Thus $G(n', p')$ contains a perfect matching convering all but at most one vertex implies a.a.s. Γ_n contains a perfect matching covering all but at most one vertex, which implies there exists a relabelling of c (using at most one more colour) which turns it into a signed colouring of $G(n,p,q)$. Note

$$p' = (1-pq)^{(2\log_b n - 2\log_b \log_b n + C')^2} = \exp\{-pq(2\log_b n - 2\log_b \log_b n + C')^2\}.$$

By Corollary 2.1.15, we know a.a.s. $G(n', p')$ contains a perfect matching a.a.s. if (ω_n is any sequence that goes to infinity)

$$\begin{aligned} p' = \exp\{-pq(2\log_b n - 2\log_b \log_b n + C')^2\} &\geq \frac{\log n' + \omega_n}{n'} \\ \iff q \leq \frac{\log(\log n' + \omega_n) - \log n'}{-p(2\log_b n - 2\log_b \log_b n + C')^2}&, \end{aligned}$$

which is true if $q \leq \frac{1}{4p\log_b n}$ for large enough n. □

In theory, we can also find n'', p'' such that a.a.s. $G(n'', p'')$ can be embedded as an induced subgraph of our auxiliary graph, such that $G(n'', p'')$ not having a perfect matching implies that our auxiliary graph also does not. However, from this to concluding that a.a.s. the signed chromatic number of $G(n, p, q)$ needs to increase, we further need union bounding over the number of optimal colourings of G^+.

Chapter 4

The Martingale Method

We are now ready to discuss the main idea behind the focus of the thesis. Recall from Proposition 2.1.11 that for $0 < p(n) < 1$ asymptotically bounded away from $0, 1$, $q = o(1)$, $G \sim G(n,p)$, $c = c(n)$ bounded,

$$r = r(n) = 2\log_b(ecn/(2\log_b n)),$$

there exists a choice of c such that a.a.s. there exists no independent set of size r in G^+. Thus there exists some constant C such that a.a.s.

$$\chi(G) \geq \frac{n}{r} = \frac{n}{2\log_b n - 2\log_b\log_b n + C}.$$

Our aim is to show that a.a.s.

$$\chi(G) \leq \frac{n}{2\log_b n - 2\log_b\log_b n + C'}$$

for another constant C'.

We will now outline the main ideas used by Bollobás in [6] to bound $\chi(G(n,p))$. Suppose that, for some r, every large enough (n^α order for some $\alpha = \alpha(n) < 1$) subgraph of G contains an independent set of size r. Then, since we can keep colouring those r-independent sets in distinct colours until there are fewer than n^α vertices left,

$$\chi(G) \leq \frac{n - n^\alpha}{r} + n^\alpha = \frac{n + (r-1)n^\alpha}{r} \leq \frac{n}{r-1}$$

for n large enough.

We want to show that we precisely have the above hold a.a.s. for $G(n,p)$. Note that a n^α-subgraph of $G(n,p(n))$ is $G(n^\alpha,p(n))$ which is simply another Erdős-Rényi graph with p asymptotically bounded away from $0,1$. Thus, if we want to apply the expected number of independent sets of size r that we computed in Proposition 2.1.11, we would be working with n^α and with

$$\begin{aligned} r(n^\alpha) &= 2\log_b n^\alpha - 2\log_b\log_b(n^\alpha) + C \\ &= 2\alpha\log_b n - 2\log_b\log_b n - 2\log_b\alpha + C. \end{aligned}$$

This looks strikingly close to

$$2\log_b n - 2\log_b\log_b n + C'$$

for another C' constant, but we need to deal with the factor of α. As we shall see in Corollary 4.1.3, it turns out that choosing $\alpha(n) = \frac{\log_b n - 1}{\log_b n}$ gets rid of α. Thus, if we can show that for *any* $\alpha(n) = \omega(\frac{1}{\log n})$ (which would encompass our choice of α above), a.a.s. every n^α-sized subgraph of $G(n,p)$ contains an independent set of size $r(n^\alpha)$, then

$$\chi(G(n,p)) \leq \frac{n}{2\log_b n - 2\log_b\log_b n + C' - 1}.$$

Note that this strategy also works for $G(n,p,q)$, by replacing independent sets of size r with special independent pairs of size $2r$. However, we will not be able to get this sharp of a bound, as we shall see in the final result Theorem 4.2.3.

A direct way to show that this containment holds a.a.s. is to show that the probability that $G(n^\alpha,p)$ ($p = p(n)$) does not have an independent set of size $r(n^\alpha)$, *union bound* over *all* choices of n^α-subgraphs in $G(n,p)$, is $o(1)$. In other words, we want to show

$$\begin{aligned} &\binom{n}{n^\alpha}\mathbf{P}\{G(n^\alpha,p) \text{ has no independent set of size } r(n^\alpha)\} \\ \leq &n^{n^\alpha}\mathbf{P}\{G(n^\alpha,p) \text{ has no independent set of size } r(n^\alpha)\} \end{aligned}$$

is $o(1)$ for any $\alpha = \omega(\frac{1}{\log n}) < 1$.

The main effort in [6] revolves around proving this bound (albeit for fixed p). Recall that the Azuma's inequality Theorem 2.3.15 concentrates the value of a martingale with bounded increments around its expectation with a very good probability. The key idea, then, is to construct a martingale $\{X_t\}_{t\geq 0}$ with bounded increments and with

$$X_N = \chi(G(n,p)),$$

such that after applying Azuma's inequality, the resulting probability is small enough that the union bound above is indeed $o(1)$.

To this end, we must first turn graph invariants into martingales. So far, in Subsection 2.3.3, we have seen a couple of examples of martingales and both are essentially sums of independent random variables. We shall now introduce a completely different way to construct martingales: in fact, any multivariate function taking random inputs can be turned into one.

Definition 4.0.1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. For $n \geq t \geq 1$, let $X_t : \Omega \to S_t$ be S_t-valued random variables (in particular S_t are topological spaces with Borel space $(S_t, \mathcal{B}(S_t))$), and let

$$f : S_1 \times \cdots \times S_n \to \mathbb{R}$$

be a Borel function. Then define

$$Y_t = Y_t(f, X_1, \ldots, X_n) := \begin{cases} \mathbf{E}\{f(X_1, \ldots, X_n)\}, & t = 0; \\ \mathbf{E}\{f(X_1, \ldots, X_n) \mid X_1, \ldots, X_t\}, & n \geq t \geq 1; \\ f(X_1, \ldots, X_n), & t > n. \end{cases}$$

$\{Y_t\}_{t \geq 0}$ is called the *Doob's martingale*.

Note that by the tower property Proposition 2.3.11 $\{Y_t\}_{t \geq 0}$ is a martingale w.r.t. $\{\mathcal{F}_t\}_{t \geq 0}$ where $\mathcal{F}_t = \sigma(X_1, \ldots, X_t)$ for $1 \leq t \leq n$, and $\mathcal{F}_t = \sigma(X_1, \ldots, X_n)$ for $t > n$. Indeed for $n \geq s \geq t$,

$$\begin{aligned} \mathbf{E}\{Y_s \mid \mathcal{F}_t\} &= \mathbf{E}\{\mathbf{E}\{f(X_1, \ldots, X_n) \mid X_1, \ldots, X_s\} \mid X_1, \ldots, X_t\} \\ &= \mathbf{E}\{f(X_1, \ldots, X_n) \mid X_1, \ldots, X_t\} = Y_t \end{aligned}$$

and note that for $t > n$, $Y_t = Y_n$.

Intuitively, this martingale is constructed from our "best guess" for the value of a function taking random inputs, having knowledge of the first t inputs. Since this is already our best guess, we should not expect this guess to increase or decrease after obtaining new information (it can increase or decrease with arbitrary probability, but the *expected* change is 0).

In the context of random graphs (either $G(n, p)$ or $G(n, p, q)$), a common and useful example of a Doob's martingale is the *edge-exposure martingale*. Here, f is any graph

invariant, and X_t are the Bernoulli variables which decide the membership (and sign) of edges in the graph.

Definition 4.0.2. Let $G(n,p)$ (resp. $G(n,p,q)$) be a random graph (resp. signed graph) with associated Bernoulli random variables X_e (resp. X_e, Y_e) for $e \in E(K_n)$ ordered. Let f be a given graph (resp. signed graph) invariant defined as a function from the set of graphs (resp. signed graphs) on n vertices to $\mathbb{R}$.

Let $N = \binom{n}{2}$. The *edge-exposure martingale* of $G(n,p)$ (resp. $G(n,p,q)$) w.r.t. the graph (resp. signed graph) invariant is the Doob's martingale $\{Y_t(f, X_{e_1}, \ldots, X_{e_N})\}_{t\geq 0}$ (resp. $\{Y_t(f, (X_{e_1}, Y_{e_1}), \ldots, (X_{e_N}, Y_{e_N})\}_{t\geq 0}$) for $1 \leq t \leq N$. We denote the edge-exposure martingale of $G(n,p)$ (resp. $G(n,p,q)$) by $Y_t(G(n,p))$ (resp. $Y_t(G(n,p,q))$).

In other words, after seeing the "status" of every edge in order, we consider the current "best guess" for the value of our graph invariant (in our case, the (signed) chromatic number). At Y_0 (i.e. when we have seen no edges), this guess is simply the expectation of the graph invariant, and at $Y_{\binom{n}{2}}$ (i.e. when we have seen every edge) this guess is the actual value of the graph invariant, as a random variable.

Note that by grouping all unused edges incident to the same vertex together into one random variable, one can similarly define a *vertex-exposure martingale*. The advantage of edge-exposure is that the random variables involved are simpler and identical, while vertex-exposure uses fewer terms before reaching the desired graph invariant (n as opposed to $\binom{n}{2}$).

Recall that the idea is to apply Azuma's inequality Theorem 2.3.15 to bound the probability that some edge-exposure martingale deviates far from its mean. A naive idea, which also serves as an example of an edge-exposure martingale, would be to simply use the *number of independent sets of size r* (resp. *number of special independent pairs of size $2r$*) as our (signed) graph invariant.

Call the martingale from the signed case $\{Y_t\}_{t\geq 0}$ and the unsigned case $\{Y'_t\}_{t\geq 0}$. Let X_r denote the number of special independent pairs of size $2r$ in $G(n,p,q)$ and let X'_r denote

the number of independent sets of size r in $G(n,p)$. Then

$$Y_t = \begin{cases} E(n,r) = \mathbf{E}\{X_r\}, & t = 0; \\ \mathbf{E}\{X_r \mid \text{existence and sign of } e_1, \ldots, e_t\}, & 1 \le t \le \binom{n}{2}; \\ X_r, & \binom{n}{2} < t. \end{cases}$$

$$Y'_t = \begin{cases} E'(n,r) = \mathbf{E}\{X'_r\}, & t = 0; \\ \mathbf{E}\{X'_r \mid \text{existence } e_1, \ldots, e_t\}, & 1 \le t \le \binom{n}{2}; \\ X'_r, & \binom{n}{2} < t. \end{cases}$$

However, recall that Azuma's inequality Theorem 2.3.15 requires that the martingale has a.s. bounded increments. For the martingales defined above, if no edge is added among the first $\binom{n}{2} - 1$, which occurs with non-zero probability, whether the last edge is there or not changes the number of independent sets on the order of $\Theta(n^{r-2})$. Thus those martingales would not work with Azuma's inequality.

A natural question, then, would be: "How do we guarantee a.s. bounded increments for a Doob's martingale?" The intuitive idea would be to simply use some f such that

$$\sup_{x_1,\ldots,x_k,x'_k,\ldots,x_n} \{|f(x_1,\ldots,x_k,\ldots,x_n) - f(x_1,\ldots,x'_k,\ldots,x_n)|\} \le c_k$$

for each $1 \le k \le n$. In other words, for each k, after fixing every other position but the k-th, the amount of variation in f from changing the k-th position is bounded by c_k (which does not depend on how the other positions are fixed).

Unfortunately, this would *not* suffice if $X_1, \ldots, X_n$ were not independent. Intuitively, one can imagine a scenario where all $X_1, \ldots, X_n$ are in fact determined by X_1, so different choices of X_1 could potentially produce vastly different $f(X_1, \ldots, X_n)$, even though for $X_2, \ldots, X_n$ fixed, varying X_1 only changes f by little. It turns out that independence on $X_1, \ldots, X_n$ is exactly what is missing, which we get for free with edge-exposure martingales.

Proposition 4.0.3. *Let $f, \{Y_t\}_{t\ge 0}, X_1, \ldots, X_n$ be as in the definition of Doob's martingale Definition 4.0.1, with $X_1, \ldots, X_N$ independent. Assume that for some $k \in \{1, \ldots, n\}$, there exists $c_k \ge 0$ constant such that we have*

$$\sup_{x_i \in S_i \forall 1 \le i \le k, x'_k \in S_k} \{|f(x_1,\ldots,x_n) - f(x_1,\ldots,x_{k-1},x'_k,x_{k+1},\ldots,x_n)|\} \le c_k$$

(x'_k is at the k-th position). Then a.s. $|Y_k - Y_{k-1}| \le c_k$.

Proof. Define

$$U := \sup_{x_k \in S_k} |\mathbf{E}\{f(X_1, \ldots, X_n) \mid X_1, \ldots, X_k\} - \mathbf{E}\{f(X_1, \ldots, x_k, \ldots, X_n) \mid X_1, \ldots, X_{k-1}\}|.$$

By definition $|Y_k - Y_{k-1}| \leq U$. Furthermore, since $X_1, \ldots, X_N$ are independent, X_k is independent from $\sigma(X_1, \ldots, X_{k-1}, f(X_1, \ldots, x_k, \ldots, X_n))$. Thus by Proposition 2.3.12

$$\mathbf{E}\{f(X_1, \ldots, x_i, \ldots, X_n) \mid X_1, \ldots, X_{k-1}\} = \mathbf{E}\{f(X_1, \ldots, x_i, \ldots, X_n) \mid X_1, \ldots, X_k\}.$$

This means

$$\begin{aligned} U &= \sup_{x_k \in S_k} |\mathbf{E}\{f(X_1, \ldots, X_n) - f(X_1, \ldots, x_i, \ldots, X_N) \mid X_1, \ldots, X_k\}| \\ &\leq \sup_{x_k \in S_k} \mathbf{E}\{|f(X_1, \ldots, X_n) - f(X_1, \ldots, x_i, \ldots, X_n)| \mid X_1, \ldots, X_k\} \\ &\leq \sup_{x_i \in E} \mathbf{E}\{c_k \mid \mathcal{F}_k\} = c_k. \end{aligned}$$

□

This means that it suffices to find a graph invariant which varies by a bounded amount whenever one edge of the graph is changed. For example, one can consider the number of independent sets of size r which share fewer than two vertices with any other independent set of size r. In that case, adding one edge at most removes one independent set, and removing one edge adds at most one independent set, so Proposition 4.0.3 applies with $c_k = 1$. The signed graph case is exactly the same, with special independent pairs of size $2r$ and with no intersection of more than one vertex between the unions of the pairs.

Let Y'_r denote the number of such "almost disjoint" independent sets (resp. Y_r and special independent pairs). A possible issue is that $\mathbf{E}\{X_r\}$ and $\mathbf{E}\{X'_r\}$ might be hard to compute and/or much smaller than $E(n, r), E'(n, r)$. We will show later that in fact

$$\begin{aligned} \mathbf{E}\{Y_r\} &= (1 - o(1))E(n, r), \\ \mathbf{E}\{Y'_r\} &= (1 - o(1))E'(n, r). \end{aligned}$$

For now, we will show that as long as that expectation is large enough (to be precise, asymptotically at least n^γ for $\gamma > 1.5$), Azuma's inequality would give us exactly the desired result, namely that we can decompose almost the entire vertex set into independent sets (or special independent pairs) of the right size. Note that the next two results apply regardless whether the graph is signed or not.

First, we directly apply Azuma's inequality to get a very strong bound on the probability that the graph invariant is small, given that it is lower bounded by some edge-exposure martingale with bounded increments whose expectation is polynomial in n.

Lemma 4.0.4. *Suppose there exists $C > 0$ constant, such that for all n, $\{Y_t\}_{t\geq 0}$ is an edge-exposure martingale with increments bounded by C. Further suppose that $Y_0 \geq (1-o(1))n^\gamma$ for some $\gamma > 0$. Set $N = \binom{n}{2}$ and let Z_n be any graph invariant such that $Z_n \geq Y_N$. Then, for any $1 > \alpha > 0$,*

$$\mathbf{P}\{Z_n \leq (1-\alpha)(1-o(1))n^\gamma\} \leq \exp\left\{-\frac{\alpha^2(1-o(1))n^{2(\gamma-1)}}{2C^2}\right\}.$$

Proof. By the Azuma-Hoeffding inequality Theorem 2.3.15, setting $a = \alpha Y_0$, we get

$$\begin{aligned}\mathbf{P}\{Z_n \leq (1-\alpha)(1-o(1))n^\gamma\} &\leq \mathbf{P}\{Y_N \leq Y_0 - a\} \leq \exp\left\{-\frac{a^2}{2\sum_{k=1}^N C^2}\right\} \\ &= \exp\left\{-\frac{\alpha^2(1-o(1))n^{2\gamma}}{2NC^2}\right\} \\ &= \exp\left\{-\frac{\alpha^2(1-o(1))n^{2(\gamma-1)}}{2C^2}\right\}.\end{aligned}$$

□

Now, we can apply this small probability to the union bound mentioned before to show that, if this graph invariant counts the number of some structure, then almost the entirety of the graph can be cut into disjoint copies of this structure.

Theorem 4.0.5. *Suppose that there exists some Z_n which denotes the number of structures H_n in $G(n,p)$ (or $G(n,p,q)$), and some $\gamma > 1.5$, for which Lemma 4.0.4 holds. Then, for any $\beta = \beta(n)$ such that $n^\beta \to \infty$ (i.e. $\beta = \omega(\frac{1}{\log n})$), a.a.s. one can find disjoint copies of H_{n^β} covering all but at most n^β vertices in $G(n,p)$ (or $G(n,p,q)$).*

Proof. Fix n and β. Note that every n^β-subgraph of G_n is G_{n^β}, so by applying Lemma 4.0.4 to n^β, we get that, with Z_{n^β} denoting the number of copies of H_{n^β} in G_{n^β} (i.e. the number of copies in a given n^β-subgraph of G_{n^β}), and with $\alpha = 0.5$, we get, for large enough n,

$$\mathbf{P}\{Z_{n^\beta} = 0\} \leq \mathbf{P}\{Z_{n^\beta} \leq 0.5(1-o(1))n^{\gamma\beta}\} \leq \exp\left\{-\frac{0.25(1-o(1))n^{2\beta(\gamma-1)}}{2C^2}\right\}.$$

Thus, union bounding over all such subgraphs of G, we get that the probability that there exists an n^β-subgraph of G without H_{n^β} is at most (let $\epsilon := \gamma - 1.5 > 0$)

$$n^{n^\beta} \exp\left\{-\frac{0.25(1-o(1))n^{2\beta(\gamma-1)}}{2C^2}\right\} = \exp\{-\Theta(n^{2\beta(\gamma-1)}) + n^\beta \log n\}$$
$$= \exp\{-\Theta(n^{\beta+2\beta\epsilon}) + n^\beta \log n\} = \exp\{-\omega(n^{\beta\epsilon})\}$$

which is $o(1)$ since $n^\beta \to \infty$. Thus a.a.s. every n^β-subgraph of G contains a copy of H_{n^β}, so by repeatedly obtaining this copy and removing it, until there are fewer than n^β vertices left, we get that a.a.s. all but a subgraph of size n^β can be decomposed into disjoint copies of H_{n^β}. □

This is what we term the *martingale method*. Assume that H_n are some induced graph structure to be counted. Then for any $0 < \beta = \beta(n)$ (which we can make converge to 1), a.a.s. all but n^β vertices of the graph can be covered by H_{n^β} as long as the following hold:

1. The count of H_n can be lower bounded by some edge-exposure martingale with bounded increments.
2. This edge-exposure martingale has expectation $\Omega(n^\gamma)$ for $\gamma > 1.5$.

Now, it remains to show that the number of "almost disjoint" independent sets (resp. special independent pairs) indeed has expectation $\Omega(n^\gamma)$, apply the martingale method, and conclude with the upper bound on the chromatic number.

4.1 Bollobás's Analysis

We start with the unsigned case, which was first done by Bollobás in 1988 [6].

Lemma 4.1.1. *Let $0 < p = p(n) < 1$ be asymptotically bounded away from $0, 1$ and $b = \frac{1}{1-p}$. Assume $r \sim 2\log_b n$ and, in $G(n,p)$, let Y'_r denote the number of independent sets S, such that $|S| = r$ and there is no other independent set of size r sharing two vertices or more with S. Then $\mathbf{E}Y'_r \sim E'(n,r)$.*

Proof. Let A_S denote the event that a given $S \subseteq V(G)$ of size r forms an independent set. For $\ell = 2, \dots, r-1$ let $Z_\ell(S)$ denote the number of independent sets of size r sharing exactly ℓ vertices with S. Finally let $Z(S) := \sum_{\ell=2}^{r-1} Z_\ell(S)$. Then Y'_r counts the number of sets S for which $\{Z(S) = 0\} \cap A_S$ holds. So

$$\begin{aligned}\mathbf{E}Y'_r &= \sum_{S \subseteq V(G)} \mathbf{P}\{A_S\}\mathbf{P}\{Z(S) = 0 \mid A_S\} \\ &= \binom{n}{r}(1-p)^{\binom{r}{2}}(1 - \mathbf{P}\{Z(S_0) \geq 1 \mid A_{S_0}\}) \\ &= E'(n,r)(1 - \mathbf{P}\{Z(S_0) \geq 1 \mid A_{S_0}\})\end{aligned}$$

with $S_0 := \{1, \dots, r\}$. Let $A := A_{S_0}$, $Z := Z(S_0)$ and $Z_\ell := Z_\ell(S_0)$ for $\ell = 2, \dots, r-1$. By Markov's inequality we know $\mathbf{P}\{Z \geq 1 \mid A_S\} \leq \mathbf{E}\{Z \mid A_S\}$. Thus

$$\mathbf{E}Y'_r = E'(n,r)(1 - \mathbf{P}\{Z \geq 1 \mid A\}) \geq E'(n,r)(1 - \mathbf{E}\{Z \mid A\}).$$

Now it suffices to show $\mathbf{E}\{Z \mid A\} = o(1)$. Note $\mathbf{E}\{Z \mid A\} = \sum_{\ell=2}^{r-1} \mathbf{E}\{Z_\ell \mid A\}$. We know

$$\mathbf{E}\{Z_\ell \mid A\} = \binom{n-r}{r-\ell}\binom{r}{\ell}(1-p)^{\binom{r}{2}-\binom{\ell}{2}},$$

as we choose ℓ vertices in the intersection, $r-\ell$ vertices outside the intersection, and all but $\binom{\ell}{2}$ (which are already missing) internal edges among the chosen vertices need to be missing. So for $2 \leq \ell \leq r-2$

$$\begin{aligned}f(\ell) :&= \frac{\mathbf{E}\{Z_{\ell+1} \mid A\}}{\mathbf{E}\{Z_\ell \mid A\}} \\ &= \frac{(n-2r+\ell)!(r-\ell)!(r-\ell)!\ell!}{(n-2r+\ell+1)!(r-(\ell+1))!(r-(\ell+1))!(\ell+1)!}(1-p)^{-\binom{\ell+1}{2}+\binom{\ell}{2}} \\ &= \frac{(r-\ell)^2}{(n-2r+\ell+1)(\ell+1)}(1-p)^{-\ell} = \frac{(r-\ell)^2}{(n-2r+\ell+1)(\ell+1)}b^\ell.\end{aligned}$$

If $\ell \leq \frac{\log_b n}{2}$ then $b^\ell \leq n^{1/2}$, while note $\frac{(r-\ell)^2}{(n-2r+\ell+1)(\ell+1)} = o(n^{-5/6})$ independent of ℓ. So $f(\ell) = o(n^{-1/3})$ for ℓ in that range. On the other hand note

$$\begin{aligned}\frac{f(\ell+1)}{f(\ell)} &= \frac{(r-\ell-1)^2(n-2r+\ell+1)(\ell+1)(1-p)^{-(\ell+1)}}{(r-\ell)^2(n-2r+\ell+2)(\ell+2)(1-p)^{-\ell}} \\ &= \left(1 - \frac{1}{r-\ell}\right)^2\left(1 - \frac{1}{n-2r+\ell+2}\right)\left(1 - \frac{1}{\ell+1}\right)\frac{1}{1-p} > 1\end{aligned}$$

as long as $1-\frac{1}{r-\ell}, 1-\frac{1}{n-2r+\ell+2}, 1-\frac{1}{\ell+1} > (1-p)^{\frac{1}{4}}$, which is true as long as n is large enough and $\frac{\log_b n}{2} < \ell < r-K$ for some constant K depending only on $\liminf_n p(n) > 0$. Thus $f(\ell)$ is increasing from $\frac{\log_b n}{2}$ to $r-K$. For all $r-K \le \ell \le r-2$, note

$$f(\ell) \ge \frac{4}{(n-r-1)(r-1)} b^{r-K} = \frac{4(1-o(1))}{b^K n \log_b n} n^{2(1-o(1))} = \omega(n^{5/6})$$

independent of ℓ (note $\limsup b(n) < \infty$). Furthermore as $f(\ell)$ is increasing between $r-K$ and $\frac{\log_b n}{2}$, $E\{Z_\ell \mid A\}$ is upper bounded in that range by the endpoints $\mathbf{E}\{Z_{r-K} \mid A\}$ and $\mathbf{E}\{Z_{(\log_b n)/2} \mid A\}$. For $1 \le k \le K$,

$$\begin{aligned}\mathbf{E}\{Z_{r-k} \mid A\} &= \mathbf{E}\{Z_{r-1} \mid A\} \prod_{\ell=2}^{k} \frac{\mathbf{E}\{Z_{r-\ell} \mid A\}}{\mathbf{E}\{Z_{r-\ell+1} \mid A\}} \\ &= \mathbf{E}\{Z_{r-1} \mid A\} \prod_{\ell=2}^{k} (f(r-\ell))^{-1} = (o(n^{-5/6}))^{k-1} \mathbf{E}\{Z_{r-1} \mid A\}.\end{aligned}$$

For $2 \le k \le \frac{\log_b n}{2}$,

$$\begin{aligned}\mathbf{E}\{Z_k \mid A\} &= \mathbf{E}\{Z_2 \mid A\} \prod_{\ell=2}^{k-1} \frac{\mathbf{E}\{Z_{\ell+1} \mid A\}}{\mathbf{E}\{Z_\ell \mid A\}} \\ &= \mathbf{E}\{Z_2 \mid A\} \prod_{\ell=2}^{k-1} f(\ell) = (o(n^{-1/3}))^{k-2} \mathbf{E}\{Z_2 \mid A\}.\end{aligned}$$

Thus

$$\mathbf{E}\{Z_\ell \mid A\} \le \begin{cases} (o(n^{-1/3}))^{\ell-2} \mathbf{E}\{Z_2 \mid A\}, & 2 \le \ell \le \frac{\log_b n}{2}; \\ \mathbf{E}\{Z_{r-K} \mid A\} + \mathbf{E}\{Z_{(\log_b n)/2} \mid A\}, & \frac{\log_b n}{2} < \ell < r-K; \\ ((o(n^{-5/6}))^{r-\ell-1} \mathbf{E}\{Z_{r-1} \mid A\}, & r-K \le \ell \le r-1. \end{cases}$$

Thus

$$\begin{aligned}&\mathbf{E}\{Z \mid A\} = \sum_{\ell=2}^{r-1} \mathbf{E}\{Z_\ell \mid A\} \\ &\le \mathbf{E}\{Z_2 \mid A\} \sum_{\ell=2}^{(\log_b n)/2} (o(n^{-1/3}))^{\ell-2} + \mathbf{E}\{Z_{r-1} \mid A\} \sum_{k=1}^{K} (o(n^{-5/6}))^{k-1} \\ &\quad + \left(r - K - \frac{\log_b n}{2}\right) \left(\mathbf{E}\{Z_2 \mid A\} (o(n^{-1/3}))^{(\log_b n)/2-2} + \mathbf{E}\{Z_{r-1} \mid A\} (o(n^{-5/6}))^{K-1}\right) \\ &\le (\mathbf{E}\{Z_2 \mid A\} + \mathbf{E}\{Z_{r-1} \mid A\})(1+o(1)).\end{aligned}$$

Finally,

$$\mathbf{E}\{Z_2 \mid A\} = \binom{n-r}{r-2}\binom{r}{2}(1-p)^{\binom{r}{2}-1} \leq n^{r-2}r^2(1-p)^{-1}n^{-2r(1-o(1))} = o(1)$$

and

$$\mathbf{E}\{Z_{r-1} \mid A\} = (n-r)r(1-p)^{\binom{r}{2}-\binom{r-1}{2}} = (n-r)r(1-p)^{r-1} = (n-r)rn^{-2(1-o(1))} = o(1)$$

so $\mathbf{E}\{Z \mid A\} = o(1)$. □

Note that for the unsigned case, similar to what we did for Proposition 2.1.11, everything would work by assuming p constant. The slightly more general case is useful in the signed case, as the positive subgraph of $G(n,p,q)$ is $G(n,p(1-q))$, and it would be hard to embed $G(n,p,q)$ as a subgraph of $G(n,p,q')$ for $q' > q$, without resorting to non-constant p. This relaxation also shows exactly what stops working when $p \to 0$ or $p \to 1$.

Recall from Proposition 2.1.11 that by manipulating the constant term in r, one can make $E'(n,r) = \omega(n^\gamma)$ for any γ. Thus the same is true for the expected number of "almost disjoint" independent sets of size r. Since the edge-exposure martingale defined by the number of such sets has bounded increments and the right expectation, the martingale method would apply here.

Theorem 4.1.2. *Let $0 < p = p(n) < 1$ be asymptotically bounded away from $0, 1$ and $b = \frac{1}{1-p}$. Let C constant be such that $C = 2\log_b(ec/2)$ where c is another constant such that $\liminf(1 - \frac{2\log c}{\log b}) > 1.5$ and let*

$$r = r(n) = 2\log_b n - 2\log_b \log_b n + C.$$

Then, for any $\beta = \omega(\frac{1}{\log n})$, a.a.s. all but n^β vertices of $G(n,p)$ can partitioned into independent sets of size $r(n^\beta)$.

Proof. By Proposition 2.1.11, $E'(n,r) = \omega(n^\gamma)$ for some $\gamma > 1.5$. Let $\{Y_t\}_{t\geq 0}$ be the edge-exposure martingale defined by f which maps a graph G to the number of independent sets of size r which intersect with no other independent set of size r at two vertices or more in G. Let Z_n denote the number of independent sets of size r in $G(n,p)$. Clearly $Z_n \geq Y_N$ $(N = \binom{n}{2})$.

Then f varies by at most 1 after switching an edge, so by Proposition 4.0.3 $\{Y_t\}$ has increments bounded by 1. Furthermore by Lemma 4.1.1 $\mathbf{E}\{Y_0\} = (1-o(1))E'(n,r) = o(n^\gamma)$, so Lemma 4.0.4 holds with Z_n counting the number of copies of independent sets of size r and $\gamma > 1.5$, so Theorem 4.0.5 applies, meaning for any $1 > \beta = \beta(n) = \omega(\frac{1}{\log n})$, a.a.s. all but n^β vertices of $G(n,p)$ can be covered by disjoint independent sets of size $r(n^\beta)$. □

Corollary 4.1.3 (Chromatic Number of $G(n,p)$)**.** *Let $0 < p = p(n) < 1$ be asymptotically bounded away from $0, 1$ and let $b = \frac{1}{1-p}$. Then there exist C_1, C_2 both constants dependent on $\limsup b$ (equivalently $\limsup p$), such that a.a.s.*

$$\frac{n}{2\log_b n - 2\log_b\log_b n + C_1} \le \chi(G(n,p)) \le \frac{n}{2\log_b n - 2\log_b\log_b n + C_2}.$$

Proof. First, by Proposition 2.1.11 and the remarks after, for $C_1 > \limsup(2\log_b(e\sqrt{b}/2))$, a.a.s.

$$\chi(G(n,p)) \ge \frac{n}{2\log_b n - 2\log_b\log_b n + C_1}.$$

Let C' constant be such that $C' = 2\log_b(ec/2)$ where c is another constant such that $\liminf(1 - \frac{2\log c}{\log b}) > 1.5$. Then by Theorem 4.1.2, for $\beta = \frac{\log_b n - 1}{\log_b n}$ (which is $\omega(\frac{1}{\log n})$), a.a.s. all but n^β vertices of $G(n,p)$ can be partitioned into independent sets of size (assume n is large enough that $\log_b n^\beta > 0$)

$$\begin{aligned} r(n^\beta) &= 2\log_b n^\beta - 2\log_b\log_b n^\beta + C' \\ &= \frac{2\log_b n(\log_b n - 1)}{\log_b n} - 2\log_b\left(\frac{\log_b n(\log_b n - 1)}{\log_b n}\right) + C' \\ &= 2\log_b n - 2\log_b(\log_b n - 1) + C' - 2 \\ &> 2\log_b n - 2\log_b\log_b n + C' - 2. \end{aligned}$$

In that case, by colouring each of those independent sets its own colour, and by colouring the remaining n^β vertices in different colours, we can have a colouring of $G(n,p)$ of at most

$$\frac{n - n^\beta}{2\log_b n - 2\log_b\log_b n + C' - 2} + n^\beta \le \frac{n}{2\log_b n - 2\log_b\log_b n + C' - 3}$$

for large enough n. Let $C_2 = C' - 3$. Then a.a.s.

$$\chi(G(n,p)) \le \frac{n}{2\log_b n - 2\log_b\log_b n + C_2}.$$

□

Notice that we could not have chosen a β constant, since that would have made $r(n^\beta)$ smaller than $r(n)$ by a constant factor, as opposed to a constant difference.

4.2 The Signed Case

Let Y_r denote the number of special independent pairs $\{H, S \setminus H\}$ in $G(n,p,q)$ of size $2r$, such that, against any other special independent pair of size $2r$ $\{H', S' \setminus H'\}$, we have

$$|S \cap S'| \leq 1$$

(note S is the union of the parts). In this case, again, an edge is contained in at most one "almost disjoint" special independent pair, so removing or adding an edge affects at most one such special independent pair.

It is worth noting that the unsigned case does *not* directly imply the signed case. Indeed in the unsigned case we showed that there are relatively few independent sets of size $r \sim 2\log_b n$ which intersect each other at more than one vertex, so that is also true in G^+. Thus one can show we mostly have $|H \cap H'|, |(S \setminus H) \cap H'|, |H \cap (S' \setminus H')|, |(S \setminus H) \cap (S' \setminus H')| \in \{0, 1, r\}$. However, from this to $|S \cap S'| \in \{0, 1, 2r\}$, we are missing a number of cases. To show $\mathbf{E}Y_r = (1 - o(1))E(n,r)$, we shall formalize the parts of the unsigned case which carry over, then prove that the intersection cases not covered by the unsigned proof are indeed rare.

Lemma 4.2.1. *Let $0 < p = p(n) < 1$ be asymptotically bounded away from $0, 1$, $b = \frac{1}{1-p}$, and $q = o(1)$. Let $c = c(n) > 0$ be such that $\log c$ is bounded and let*

$$r = r(n) = 2\log_b(ecn/(2\log_b n)) \sim 2\log_b n.$$

Let $q \geq C/\log n$ for some $C > 0$ large dependent on the asymptotic bounds on $p, \log c$. Let Y_r denote the number of special independent pairs $\{H, S \setminus H\}$ in $G(n,p,q)$ of size $2r$, such that $|S \cap S'| < 2$ for any other $\{H', S' \setminus H'\}$ special independent pair in $G(n,p,q)$ of size $2r$. Then $\mathbf{E}Y_r \sim E(n,r)$.

Proof. We do a set-up analogous to the proof of Lemma 4.1.1. For convenience, we will call special independent pairs of size $2r$ *hit pairs*. Given $H, S \setminus H \subseteq V(G)$, both of size r, with H having a smaller first vertex than $S \setminus H$, let $A_{H,S}$ denote the event that $\{H, S \setminus H\}$ is a

hit pair. Let $Z(H,S)$ denote the number of other hit pairs whose union shares two or more vertices with S. Then Y_r counts the number of hit pairs $\{H, S \setminus H\}$ for which $Z(H,S) = 0$. In other words

$$\mathbf{E}Y_r = \sum_{H,S} \mathbf{P}\{A_{H,S}\}\mathbf{P}\{Z(H,S) = 0 \mid A_{H,S}\} \geq \sum_{H,S} \mathbf{P}\{A_{H,S}\}(1 - \mathbf{E}\{Z(H,S) \mid A_{H,S}\}).$$

Clearly $\mathbf{E}\{Z(H,S) \mid A_{H,S}\}$ does not depend on H, S (by symmetry). Thus let A denote the event that $H = \{1, \ldots, r\}, S \setminus H = \{r+1, \ldots, 2r\}$ is a hit pair and let Z denote the number of other hit pairs whose union shares two or more vertices with $\{1, \ldots, 2r\}$. Furthermore $\sum_{H,S} \mathbf{P}\{A_{H,S}\} = E(n,r)$ by definition. Then

$$\mathbf{E}Y_r \geq E(n,r)(1 - \mathbf{E}\{Z \mid A\}).$$

Thus it suffices to show that $\mathbf{E}\{Z \mid A\} = o(1)$. For every $0 \leq i, j, k, \ell \leq r$, let $Z_{i,j,k,\ell}$ denote the number of hit pairs $\{H', S' \setminus H'\} \neq \{\{1, \ldots, r\}, \{r+1, \ldots, 2r\}\}$, with H' having the least first element, such that

$$|H' \cap \{1, \ldots, r\}| = i, |(S' \setminus H') \cap \{1, \ldots, r\}| = j,$$
$$|H' \cap \{r+1, \ldots, 2r\}| = k, |(S' \setminus H') \cap \{r+1, \ldots, 2r\}| = \ell.$$

Since these are counts of independent sets intersecting in G^+ the positive subgraph of $G(n,p,q)$, from the proof of the unsigned case Lemma 4.1.1 applied to G^+ (which is $G(n, p(1-q))$), we know that $\sum_{2 \leq i \leq r-1} \mathbf{E}\{Z_{i,j,k,\ell} \mid A\} = o(1)$ and same for the sums over j, k, ℓ. This means we only need to consider when each $i, j, k, \ell \in \{0, 1, r\}$.

Thus we actually only have the case where one of i, k, ℓ is r (j cannot be r, and i, ℓ being both r means $\{H', S' \setminus H'\} = \{\{1, \ldots, r\}, \{r+1, \ldots, 2r\}\}$), and the case $0 \leq i, j, k, \ell \leq 1$. Call $\tilde{Z}$ the sum over the first case and Z' the sum over the second.

In the first case, there are 2 choices for the set that is equal to $\{1, \ldots, r\}$ or $\{r+1, \ldots, 2r\}$ (call this one H_1), and fewer than $\binom{n}{r}$ choices for the other set (call this H_2). Note H_2 has at most one vertex in $\{1, \ldots, 2r\}$. For a given $\{H_1, H_2\}$ to be a hit pair, H_2 must miss $\binom{r}{2}$ positive edges and H_1, H_2 must miss at least $r(r-1)$ negative edges between them. These sets of edges are disjoint and they are disjoint from the edges affected by A. Thus

$$\begin{aligned}\mathbf{E}\{\tilde{Z} \mid A\} &< 2\binom{n}{r}(1 - p(1-q))^{\binom{r}{2}}(1-pq)^{r(r-1)} \\ &= 2\mathbf{E}\{\text{Number of independent sets in } G^+ \text{ of size } r\}(1-pq)^{r(r-1)}.\end{aligned}$$

We know from Proposition 2.1.11 that $\mathbf{E}\{\text{Number of independent sets in } G^+ \text{ of size } r\} = O(n^\alpha)$ for some constant α (dependent on the asymptotic bounds on p and c, but crucially not on C). However

$$(1-pq)^{r(r-1)} \leq \exp\{-\frac{2\log n}{\log b} pqr(1-o(1))\} = n^{-2(1-o(1))pqr/\log b}$$

and $2pqr/\log b \geq 4pC(1-o(1))/(\log b)^2 > \alpha$ for large enough n, if $C > \alpha(\limsup \log b)^2/4p$. Thus $\mathbf{E}\{\tilde{Z} \mid A\} = o(1)$.

In the second case, there are no more than 16 choices of i, j, k, ℓ. Given a choice of i, j, k, ℓ, there are fewer than $\binom{n}{r}^2$ choices for $H, S \setminus H$. For a given $\{H, S \setminus H\}$ to count, each must miss $\binom{r}{2}$ positive edges and they must miss at least $(r-1)^2$ negative edges between them. Again these sets of edges are disjoint and unaffected by A. Thus

$$\begin{aligned}\mathbf{E}\{Z' \mid A\} &< 16\binom{n}{r}^2 (1-p(1-q))^{2\binom{r}{2}}(1-pq)^{(r-1)^2} \\ &= 16(\mathbf{E}\{\text{Number of independent sets in } G^+ \text{ of size } r\})^2(1-pq)^{(r-1)^2}.\end{aligned}$$

Similar to the previous case, $(\mathbf{E}\{\text{Number of independent sets in } G^+ \text{ of size } r\})^2 = O(n^{2\alpha})$, and

$$(1-pq)^{(r-1)^2} \leq n^{-2(1-o(1))pqr/\log b},$$

with $2pqr/\log b > 2\alpha$ for n large enough, if $C > 2\alpha(\limsup \log b)^2/4p$ (which also satisfies the previous bound on C). So $\mathbf{E}\{Z' \mid A\} = o(1)$.

All in all

$$\mathbf{E}\{Z \mid A\} = \mathbf{E}\{\tilde{Z} \mid A\} + \mathbf{E}\{Z' \mid A\} + o(1) = o(1). \qquad \square$$

Notice that compared to the unsigned case Lemma 4.1.1, our Lemma Lemma 4.2.1 introduced a restriction on q. Intuitively, having $q = 0$ and the fact that many independent sets of size r in $G^+ \sim G(n, p(1-q))$ are disjoint (which allowed our colouring in the unsigned case) would mean that any pair of them can form a special independent pair of size $2r$, causing a lot of overlap.

Now, exactly the same as in the unsigned case, we have an edge-exposure martingale with unit increments, with expectation polynomial in n, such that its N-th term ($N = \binom{n}{2}$) lower bounds the number of special independent pairs of size r in $G(n, p, q)$. The martingale method applies in exactly the same way here.

Theorem 4.2.2. *Let $0 < p = p(n) < 1$ be asymptotically bounded away from $0, 1$, $q = o(1)$, and $b = \frac{1}{1-p}$. Let C constant be such that $C = 2\log_b(ec/2)$ where c is another constant such that $\liminf 2(1 - \frac{2\log c}{\log b}) > 1.5$ and let*

$$r = r(n) = 2\log_b n - 2\log_b \log_b n + C.$$

Assume $q \geq \frac{C'}{\log n}$ for some constant C' large enough (dependent on C and on the asymptotic bounds on p) so that Lemma 4.2.1 applies. Then, for any $\beta = \omega(\frac{1}{\log n}) \leq 1$, a.a.s. all but n^β vertices of $G(n, p, q)$ can be partitioned into special independent pairs of size $2r(n^\beta)$.

Proof. By Proposition 3.1.3, due to our choice of C, $E(n, r) = \omega(n^\gamma)$ for some $\gamma > 1.5$. Let f be a function which maps a signed graph G to the number of special independent pairs of size $2r$ which intersect with no other special independent pair of size $2r$ at two vertices or more in G. For each $n \geq 1$, let $\{Y_t\}_{t\geq 0}$ be the edge-exposure martingale defined by f on $G(n, p, q)$. Let Z_n denote the number of special independent pairs of size $2r$ in $G(n, p, q)$. Clearly $Z_n \geq Y_N$ $(N = \binom{n}{2})$.

Also, f varies by at most 1 after switching an edge, so by Proposition 4.0.3 $\{Y_t\}$ has increments bounded by 1. Furthermore by Lemma 4.2.1 $\mathbf{E}\{Y_0\} = (1 - o(1))E(n, r) = o(n^\gamma)$, so Lemma 4.0.4 holds with Z_n counting the number of copies of special independent pairs of size $2r$ and $\gamma > 1.5$, so Theorem 4.0.5 applies, meaning for any $1 > \beta = \beta(n) = \omega(\frac{1}{\log n})$, a.a.s. all but n^β vertices of $G(n, p, q)$ can be covered by disjoint special independent pairs of size $r(n^\beta)$. □

Since the main work of the proof was done in Lemma 4.2.1, and in the martingale method Lemma 4.0.4 and Theorem 4.0.5, the main underlying ideas are the same as the proof for the unsigned version Theorem 4.1.2. The main difference lies in the restriction on q, since our Lemma Lemma 4.2.1 does not apply to small q.

Note that by Corollary 4.1.3, the positive subgraph of $G(n, p, q)$ has chromatic number approximately

$$\frac{n}{2\log_{b'} n - 2\log_{b'} \log_{b'} n + C},$$

where $b' = \frac{1}{1-p(1-q)}$. This chromatic number will serve as the lower bound for the signed chromatic number of $G(n, p, q)$, while all our results on its upper bound stem from the expected number of special independent pairs Proposition 3.1.3, which uses $b = \frac{1}{1-p} > b'$.

Due to that, as well as the issue of having to treat q by case, our concluding corollary will have to be slightly weaker than the unsigned version.

Theorem 4.2.3 (Chromatic Number of $G(n,p,q)$)**.** *Let $0 < p = p(n) < 1$ be asymptotically bounded away from $0, 1$, $q = o(1)$, and $b = \frac{1}{1-p}$. Then there exist functions $b_1 = b_1(n)$ and $b_2 = b_2(n)$, such that $b_1 \sim b_2 \sim b$, and a.a.s.*

$$\frac{n}{2\log_{b_1} n} \leq \chi(G(n,p,q)) \leq \frac{n}{2\log_{b_2} n}.$$

Proof. First, by Proposition 2.2.6 and by Theorem 4.1.2, we know that for some C_1 dependent on $\limsup b$ (in fact $C_1 > \limsup(2\log_b(e\sqrt{b}/2))$), a.a.s.

$$\chi(G(n,p,q)) \geq \chi(G(n,p(1-q))) \geq \frac{n}{2\log_{b'} n - 2\log_{b'}\log_{b'} n + C_1}$$

for $b' = \frac{1}{1-p(1-q)}$, so the left inequality in the statement is true if we choose $b_1 \sim b' \sim b$ such that

$$\frac{2\log n}{\log b_1} = 2\log n\left(\frac{1}{\log b'} - \frac{\log_{b'}\log_{b'} n}{\log n} + \frac{C_1}{2\log n}\right).$$

For b_2, let C' constant be such that $C' = 2\log_b(ec/2)$ where c is another constant such that $\liminf 2(1 - \frac{2\log c}{\log b}) > 1.5$. For now, we first assume $q > \frac{C''}{\log n}$ where C'' large enough (dependent on C' and on asymptotic bounds on p, so ultimately on asymptotic bounds on p) is such that Theorem 4.2.2 applies. Then by Theorem 4.2.2, for $\beta = \frac{\log_b n - 1}{\log_b n}$ (which is $\omega(\frac{1}{\log n})$), a.a.s. all but n^β vertices of $G(n,p)$ can be partitioned into special independent pairs of size $2r(n^\beta)$, where (assume n is large enough that $\log_b n^\beta > 0$)

$$\begin{aligned} r(n^\beta) &= 2\log_b n^\beta - 2\log_b\log_b n^\beta + C' \\ &= \frac{2(\log_b n)(\log_b n - 1)}{\log_b n} - 2\log_b\left(\frac{(\log_b n)(\log_b n - 1)}{\log_b n}\right) + C' \\ &= 2\log_b n - 2\log_b(\log_b n - 1) + C' - 2 \\ &> 2\log_b n - 2\log_b\log_b n + C' - 2. \end{aligned}$$

In that case, by colouring each of those special independent pairs with two opposite colours, and by colouring the remaining n^β vertices in colours of distinct absolute value, we can

have a colouring of $G(n,p,q)$ of at most

$$\frac{n-n^\beta}{2\log_b n - 2\log_b\log_b n + C' - 2} + 2n^\beta \leq \frac{n}{2\log_b n - 2\log_b\log_b n + C' - 3}$$

for large enough n. Similar to b_1, the right inequality in the statement is true if we choose $b_2 \sim b$ such that

$$\frac{1}{\log b_2} = \frac{1}{\log b} - \frac{\log_b\log_b n}{\log n} + \frac{C'-3}{2\log n}.$$

Finally, we assume $q < \frac{C''}{\log n}$. Note that $G(n,p,q)$ can be embedded as a subgraph of

$$H \sim G\left(n, p(1-q+\frac{C''}{\log n}), \frac{C''}{\log n}\right),$$

where $G(n,p,q)$ has the same positive subgraph, but (possibly) fewer negative edges. Since $\chi(G(n,p,q)) \leq \chi(H)$, we can use the same b_2 obtained from applying the previous case to H. □

Note that this result is weaker than its unsigned counterpart Corollary 4.1.3. This is because there is a mismatch between the log base of the best known lower bound, $\frac{1}{1-p(1-q)}$ used by $\chi(G^+)$, and the log base used by the a.a.s. largest special independent pairs in $G(n,p,q)$, $\frac{1}{1-p}$. Nonetheless, we still have that for all $\epsilon > 0$, a.a.s.

$$\frac{n(1-\epsilon)}{\log_b n} < \chi(G(n,p,q)) < \frac{n(1+\epsilon)}{\log_b n},$$

or, a.a.s.

$$\chi(G(n,p,q)) \sim \frac{n}{\log_b n} \sim \chi(G(n,p)) \sim \chi(G(n,p(1-q))$$

for any p asymptotically bounded away from $0,1$, $b = \frac{1}{1-p}$, and $q = o(1)$. In other words, adding few negative edges or labelling few edges negative in a $G(n,p)$ graph a.a.s. changes its chromatic number by less than any constant factor.

It is worth noting that when $q < \frac{1}{4p\log_b n}$, by applying Proposition 3.2.1, we fall into the $\chi(G(n,p,q)) = \chi(G^+)$ case, so we get back the stronger result from Corollary 4.1.3.

Chapter 5

Conclusion

Let us first give a summary of the results. We showed that a.a.s. the signed chromatic number of random signed graphs is asymptotically the same as its trivial lower bound, namely the chromatic number of its positive subgraph, when there are few negative edges.

To prove this, we first defined a $G(n,p,q)$ model of random signed graphs based on the $G(n,p)$ model of random graphs, and slightly modified from [17]. Knowing from [6] that

$$\chi(G(n,p)) \sim \frac{n}{2\log_b n},$$

where p is constant and $b = \frac{1}{1-p}$, we showed that

$$\chi(G(n,p,q)) \sim \frac{n}{2\log_b n},$$

where p is constant (actually, it suffices to be asymptotically bounded away from $0,1$), $b = \frac{1}{1-p}$ and $q = o(1)$.

To prove our result, we defined a signed analogue to independent sets, termed *special independent pairs*. We showed that we expect many such pairs of size asymptotically $2\log_b n$ on each half of the pair. Before proceeding further, we also proved that for q small enough (less than $\frac{1}{4p\log_b n}$)), a.a.s. any optimal unsigned colouring of the positive subgraph also works for the signed graph, by matching the colour classes in a perfect matching, where the matched colour classes have no negative edges between them, and by relabelling the matched colours into opposites.

At this point, we adapt the proof used by Bollobás [6] on $G(n,p)$ to $G(n,p,q)$. It would suffice to prove that the probability that $G(n,p,q)$ contains *one* such pair is so large, that

by union bound, a.a.s. any "large" (sub-1 power of n sized) subgraph of $G(n,p,q)$ contains such a special independent pair. Indeed, if that were true, then we can keep picking out special independent pairs and colouring their halves in opposite colours, until there are n^α vertices left to be coloured individually, where $0 < \alpha < 1$.

To prove that that probability is indeed large, we use *Azuma's inequality* [4] [14], which would apply as long as we can phrase the number of such special independent pairs as the value of some martingale with *bounded increments*. Simply looking at the edge-exposure martingale for the number of pairs is not enough, since changing one edge can change the expected number by a lot. Thus we look at the number of "almost" disjoint special independent pairs (of the right size), whose conditional expected number indeed changes by at most 1 when one edge is added or removed. We prove that this substitution works, i.e. the number of such "almost" disjoint special independent sets is asymptotically the same as without the "almost" disjoint condition, when q is not too small.

Finally, we apply the inequality and check that everything indeed works out. Compared to the unsigned case, particular care was necessary in dealing with different ranges of q.

5.1 Future Work

There are a couple of avenues to improve our result. First, our starting point was the belief that, for small q, since the expected number of large special independent pairs behaves the same as that of independent sets, so should the signed chromatic number be similar to the chromatic number. This would no longer be true for q away from 0. In that case, including the case where $q = \frac{1}{2}$ (the uniformly random signed graph), the signed chromatic number of $G(n,p,q)$ remains unexplored.

Second, we have proven Proposition 3.2.1 that for small enough q ($q < \frac{1}{4p\log_b n}$), the signed chromatic number is unchanged before and after adding negative edges. However, it is unknown whether this is still true for $q \gg \frac{1}{\log n}$. If the chromatic number *is* unchanged, then there must exist some unsigned colouring such that the auxiliary graph admits a perfect matching. However, recall that by 4.2.1, for slightly larger q ($q > \frac{C}{\log n}$ for some C constant), the special independent pairs are mostly disjoint, which means there should be few options to match up independent sets. In addition, the probability that a specific unsigned colouring admits a perfect matching in the auxiliary graph diminishes rapidly without much increase in q. Therefore, we conjecture that $\frac{1}{\log n}$ is actually the threshold for the chromatic number being unchanged after adding negative edges.

Finally, there is a gap between the a.a.s. lower and upper bounds of $\chi(G(n,p,q))$ when Proposition 3.2.1 does not apply. The lower bound uses the log base $\frac{1}{1-p(1-q)}$ while the upper bound uses at least $\frac{1}{1-p}$. We believe the upper bound to be closer to the truth, since $\frac{1}{1-p}$ is the base used by the size of the maximal special independent pairs, but there is nothing that dictates that the signed colour classes *must* look like special independent pairs. Currently, it is not known whether $\chi(G(n,p,q)) > \chi(G(n,p(1-q)))$ for *any* range of q. If the lower bound can be improved for larger q, then this will also be progress towards the previous avenue of research.

5.1.1 Generalization to Voltage Graphs

Our result also has a natural avenue for generalization. Voltage graphs, first defined by Gross in 1974 [12], generalize signed graphs so that edge labels are now oriented group elements from an arbitrary group.

Definition 5.1.1. A *voltage graph* (also *gain graph*) G is a digraph (V,E) equipped with a *voltage*

$$\alpha : E \to X,$$

where X is a group.

The only meaning of the edge orientations is to make the group labels directional. Suppose (u,v) is labelled g. Then (v,u), despite not being an arc, is understood to be labelled g^{-1}. Signed graphs can be seen as voltage graphs where $X = \mathbb{Z}_2$, in which case all elements are self-inverse so edge orientations are unnecessary. When the voltage X is a finite group, we can readily extend signed colouring to voltage graphs.

Definition 5.1.2. Let G be a voltage graph whose voltage has order n. A *(proper n-multiple) voltage (vertex) colouring* of G in nk colours is a map

$$c : V(G) \to \{(g,i) : g \in X, i \in \{1,\dots,k\}\},$$

where $g = g(v)$ is the *colour voltage* and $i = i(v)$ is the *colour value* at v, such that for all $(u,v) \in E(G)$, if $i(u) = i(v)$, then

$$g(u) \neq \alpha(u,v)g(v).$$

The *voltage chromatic number* of G is the minimal nk for which such a c exists.

One can verify that this definition is consistent even when considering $\alpha(v,u)$:

$$g(u) \neq \alpha(u,v)g(v) \iff \alpha(v,u)g(u) = (\alpha(u,v))^{-1}g(u) \neq g(v).$$

We can also extend vertex switching.

Definition 5.1.3. Let G be a voltage graph with finite voltage, $v \in V(G)$ and $x \in X$. An *x-vertex switch* at v is a voltage graph with the same vertex and arc sets as G, but the label of all arcs incident to v are left-multiplied by x.

One can verify that given a voltage colouring of G, by left-multiplying the colour voltage of v by x, we get a voltage colouring of the x-switched graph:

$$g(u) \neq \alpha(u,v)g(v) \iff xg(u) \neq x\alpha(u,v)g(v).$$

Finally, in order to extend the Erdős-Rényi model to voltage graphs, it would be natural to require that, for every $g \in X$, the subgraph consisting of arcs labelled g to be an Erdős-Rényi random graph when viewed as an undirected graph. However, there are different ways to actually orient the labels. For signed graphs, where orientations do not matter, $G(n,p,q)$ was the only sensible model, and we found out that a.a.s. the signed chromatic number is asymptotically the same before and after adding negative edges. We conjecture that the same is true for voltage graphs no matter the orientation. In other words, we conjecture that, as long as the undirected graph consisting of g-labelled arcs follows $G(n,p_g)$ for $p_g = o(1)$ if $g \neq \mathbf{1}$ ($\mathbf{1}$ being the identity) and $p_{\mathbf{1}}$ is asymptotically bounded away from $0, 1$, the voltage chromatic number is a.a.s. asymptotically equal to $\chi(G(n,p_{\mathbf{1}}))$.

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