Laconic Evaluation of Branching Programs from the Diffie-Hellman Assumption

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

This thesis is based on the paper "Laconic Evaluation of Branching Programs from the Diffie-Hellman Assumption" co-authored by Sanjam Garg, Mohammad Hajiabadi, Peihan Miao, and Alice Murphy [GHMM24a]. The aforementioned paper appeared at the International Association for Cryptologic Research (IACR) Public Key Cryptography (PKC) conference in April 2024 [GHMM24b]. The copyright for [GHMM24b] is held by the IACR. https://doi.org/10.1007/978-3-031-57725-3_11.

Murphy is the sole author of all differences between this work and [GHMM24a]. Authorship is as follows.

- Murphy authored the abstract.
- Hajiabadi, Miao, and Murphy were the primary authors of Chapter 1: Introduction and Chapter 2: Technical Overview.
- Chapter 3: Preliminaries consists of standard definitions. Adaptations and elaborations of the definitions were authored by Murphy.
- Chapter 4: Semi-Honest Laconic 2PC with Branching Programs contains Construction 1 for BP-2PC and its security proof, which are built off of Construction 1 for ℓPSI and its associated security proof from [ABD+21]. All differences were authored by Murphy with guidance from Hajiabadi. The correctness proof for Construction 1 was authored by Murphy.
- Chapter 5: Applications was authored by Murphy.

Abstract

Secure two-party computation (2PC) enables two parties to compute a function f on their joint inputs while keeping their inputs private. Laconic cryptography is a special type of 2PC in which this is done with asymptotically optimal communication in only two rounds of communication. The party who sends the first message is called the *receiver* and the party who replies with the second message is called the *sender*. Laconic cryptography considers the case of asymmetric input sizes, where the receiver's input is much larger than the sender's input or vice versa. As such, the size of the messages sent cannot depend on the size of the larger input. For example, if x_R is the receiver's input, x_S is the sender's input, and $|x_R| \gg |x_S|$, then the protocol's communication cost cannot depend on $|x_R|$, but it may depend on $|x_S|$.

Previous works have shown protocols can be built for laconic oblivious transfer (OT) [Cho *et al.* CRYPTO 2017] and laconic private set intersection (PSI) [Alamati *et al.* TCC 2021] from the Diffie-Hellman assumption. Quach, Wee, and Wichs [FOCS 2018] give a construction for laconic 2PC for *general functionalities* based on the Learning with Errors (LWE) assumption. In this work, we bridge the gap by giving a laconic protocol for the evaluation of branching programs (BPs) from the Diffie-Hellman assumption. In this setting, the receiver holds a large branching program BP and the sender holds a short input x. Our protocol allows the receiver to learn x if and only if $\mathrm{BP}(x) = 1$, and nothing more. The communication cost only grows with the size of x and the depth of BP, and does not further depend on the size of BP. Our construction can be used to realize PSI and private set union (PSU) functionalities and can handle unbalanced BPs and BPs with wildcards.

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List of Abbreviations

2PC two-party computation. 1, 3

BP branching program. 3–5, 15

CDH computational Diffie-Hellman. 3, 4, 6, 9, 12, 14

CDS conditional disclosure of secrets. 12

DCR decisional composite residuosity. 3

DDH decisional Diffie-Hellman. 3

FHE fully homomorphic encryption. 3

HE hash encryption. 8

LWE learning with errors. 3, 4, 6, 9, 12

MPC multi-party computation. 1

OT oblivious transfer. 2, 3, 5

PPT probabilistic polynomial time. 14

PSI private set intersection. 1, 4, 5, 44

PSU private set union. 7, 44

QR quadratic residuosity. 3

Chapter 1

Introduction

Secure two-party computation (2PC) allows two parties, each holding a private input, to jointly compute the output of a function while hiding all other information about their inputs. This is a special case of secure multi-party computation (MPC) [Yao86, GMW87].

A famous example of 2PC is Yao's Millionaires' Problem [Yao82] in which millionaires Alice and Bob wish to learn who has more money, but neither wishes to divulge how much money they have. In this case, the function f they wish to jointly compute is the truth value of $x_a > x_b$, where x_a is Alice's wealth and x_b is Bob's wealth. The output of $f(x_a, x_b)$ is a single bit: 1 if Alice is richer; 0 otherwise. The computation of this bit must not reveal any additional information about x_b to Alice, nor any additional information about x_a to Bob. One type of 2PC we consider is private set intersection (PSI). In this setting, Alice and Bob each hold a set of elements and wish to securely, and efficiently, learn the intersection of their sets. Then, $f(A, B) = A \cap B$, where A and B are Alice and Bob's sets of elements, respectively.

At a minimum, such a protocol requires two rounds of communication. Alice sends a message to Bob and Bob sends a message in reply. This information should allow Alice to learn the output of $f(A, B)$. For correctness, Alice's message must depend on A and Bob's message must depend on B. Security requires that these messages hide their dependent value. Minimizing the number of rounds to two is advantageous because it reduces the time parties must spend online waiting for their 'turns'. Of course, generally, for efficiency, we also want to minimize the message sizes and the computational costs. However, depending on the setting, a protocol might minimize the communication and computation for one party, while letting the other bear the brunt of the costs. Consider the following PSI example.

[ABD+21] coined the term "self-detecting encryption" for a scheme that can determine if the encryption payload is in an "illegal" set, while maintaining security if the plaintext is legal. In this setup, a third party will make publicly available the (possibly large) database of hashes of the illegal messages. Applications for this detection system include firewalls to block access to certain websites and cloud storage systems to detect if known illegal content (e.g. abuse material) has been uploaded. In this setup, the firewall or cloud storage system plays the role of Alice with a large input and the user trying to access a website or upload content to the cloud plays the role of Bob with a much smaller input.

Setup Variants. There are variations of the problem setup described above which can have a significant impact on how the problem is solved. Table 1.1, which is described below, summarizes some of these differences. In the two-round setting, the party who sends the first message learns the output of the joint computation. This is often called the *receiver* party, and the other is called the *sender*.

Table 1.1: Summary of 2PC constructions for different functionalities and relative input sizes.

Inputs	**Functionality**	**Assumptions**	**Total communication cost**	**Works**
A, B	General f	Garbled circuits + OT	At least $\|A\| + \|B\|$	
$\|A\| \ll \|B\|$	General f	LWE	$O(\|f(A,B)\| + \|A\|)$	[GSW13]
	The BP B	DCR, DDH, QR, or LWE	$O(\text{BP depth})$	[IP07]
$\|A\| \gg \|B\|$	General f	LWE	$O(\|f(A,B)\| + \|B\|)$	[QWW18]
	Laconic OT	DDH, CDH, or QR	$O(\|\text{OT output}\|)$	[CDG+17]
	Laconic PSI	CDH or LWE	$O(\|B\| \log \|A\|)$	[ABD+21]
		Pairings	$O(\|B\|)$	[ALOS22]
	Bit-checking BP A	CDH or LWE	$O(\|B\|, \text{BP depth})$	This work

As shown in the first row of Table 1.1, garbled circuits [Yao86] together with oblivious transfer (OT) [Rab05, Rab81] enables 2PC for general functionalities f with two rounds of communication—one message from the *receiver* to the *sender* and another message from the sender back to the receiver. This approach achieves the optimal round complexity; nevertheless, it requires the communication complexity to grow with the size of f. In particular, if we represent f as a Boolean circuit, then the communication grows with the number of gates in the circuit, which grows at least with the size of the inputs A and B.

For unbalanced input lengths (i.e., $|A| \gg |B|$ or $|A| \ll |B|$), *is it possible to make the communication only grow with the shorter input and independent of the longer input?*

Long Sender Input. When the sender has a long input, i.e. $|A| \ll |B|$, we can use fully homomorphic encryption (FHE) [Gen09] to achieve communication that only grows with the receiver's input length $|A|$ plus the function output length. This technique works for any function but can only be based on variants of the learning with errors (LWE) assumption [GSW13]. For simpler functions that can be represented by a branching program (BP), in particular, if the sender holds a private large branching program BP and the receiver holds a private short input A, the work of Ishai and Paskin [IP07] illustrates how to construct 2PC for $BP(A)$ where the communication only grows with $|A|$ and the *depth* of BP, and does not further depend on the size of BP. Their construction is generic from a primitive called *rate-1 OT*, which can be built based on a variety of assumptions such as decisional composite residuosity (DCR), decisional Diffie-Hellman (DDH), quadratic residuosity (QR), and LWE with varying efficiency parameters [IP07, DGI+19, GHO20, CGH+21]. In this setting, there are works in secure branching program and decision tree evaluation for applications in machine learning and medicine [BPSW07, BFK+09, KNL+19, CDPP22]. However, our results concern the dual setting (described below), in which the receiver has the longer input and learns the output in only two rounds of communication.

Long Receiver Input. When the receiver has a long input, i.e. $|A| \gg |B|$, a recent line of work on *laconic cryptography* [CDG+17, QWW18, DGGM19, ABD+21, ALOS22] focuses on realizing secure 2PC with asymptotically-optimal communication in two rounds. In particular, the receiver has a large input and the size of her outgoing message only depends on the security parameter and not her input size. The second message (sent by the sender) as well as the sender's computation may grow with the size of the sender's input, but should be independent of the receiver's input size.

In this dual setting, the work of Quach, Wee, and Wichs [QWW18] shows how to realize laconic 2PC for general functionalities using LWE. Regarding laconic 2PC for simpler functions from assumptions other than LWE, much less is known compared to the setting of long sender inputs.

Cho et al. [CDG+17] introduced the notion of laconic oblivious transfer (laconic OT), where the receiver holds a large input $D \in \{0,1\}^n$, the sender holds an input $(L \in [n], m_0, m_1)$, and a two-round protocol allows the receiver to learn $(L, m_{D[L]})$ and nothing more. The communication complexity as well as the sender's computation only grow with the security parameter and is independent of the size of D. Besides LWE [QWW18], laconic OT can be built from DDH, computational Diffie-Hellman (CDH), or QR assump-

tions [CDG+17, DG17].[1] Recent work [ABD+21, ALOS22] extends the functionality to laconic private set intersection (laconic PSI), where the sender and receiver each hold a private set of elements B and A respectively ($|A| \gg |B|$), and the two-round protocol allows the receiver to learn the set intersection $A \cap B$ and nothing more. The communication complexity and the sender's computational complexity are both independent of the larger set $|A|$. Laconic PSI can be built from CDH, LWE [ABD+21], or pairings [ALOS22].

BPs. In this work, we consider laconic 2PC for functionalities represented by branching programs. Consider 2PC where the receiver's input is a branching program and the sender holds a set as input. The receiver's branching program is a directed binary tree with each internal node encoding a bit index and each leaf encoding either 1 for "accept" or 0 for "reject". Edges from a parent node to a left (respectively right) child are labelled 0 (resp. 1). The BP can be evaluated on bit string inputs starting at the root. The value of an input string at the index encoded in the root determines whether the evaluation path proceeds to the left (bit value 0) or right (bit value 1). At the next node, the same bit-checking evaluation is done with respect to the encoded index. In this way, the input string induces a root-to-leaf path down the tree. If the terminal leaf encodes 1, the input string is in the receiver's set; otherwise, it is not in the receiver's set. The branching program may be unbalanced, with root-to-leaf paths of varying lengths.

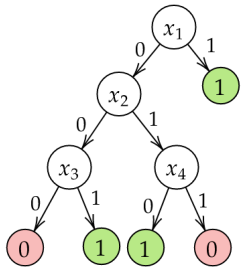


Figure 1.1: **Bit-checking branching program describing a set.**

For an example, consider the unbalanced BP in Figure 1.1. This BP can be evaluated on inputs in $\{0,1\}^4$. To illustrate the evaluation, suppose we compute $BP(x)$ with $x' = 0010$. The root node encodes index 1. Since $x'[1] = 0$, we travel to the left child, which encodes index 2. Since $x'[2] = 0$, we travel again to the left child, which encodes index 3. Since $x'[3] = 1$, we travel to the right child, which is a leaf node encoding 1. The evaluation then outputs 1, which indicates "accept".

[1] Importantly, in laconic OT, the receiver's second-phase computation time should have at most a polylog dependence on $|D|$. This can be achieved in the laconic OT setting because the index i is known to the receiver. In the other settings, such as laconic PSI, this cannot be realized (without pre-processing) because *not* probing a particular database entry leaks information about the sender's input.

PSI WITH A BP. Suppose the receiver holds a branching program, BP, with a polynomial number of root-to-leaf paths that represents a potentially large set of elements, A. An element x is in A if $BP(x) = 1$ and not in A if $BP(x) = 0$. The BP is evaluated on input x by following the path induced by x in the BP to a leaf node, then outputting the bit encoded in the leaf. The receiver wants to send a succinct message to an untrusted sender party so that the sender can reply with another succinct message that allows the receiver to learn $A \cap B$, where B is a small set held by the sender. For security, the receiver must not be able to learn anything about B beyond the intersection and the sender must not be able to learn anything about A.

LACONIC OT WITH A BP. Suppose the receiver holds a potentially large database, $D \in \{0,1\}^n$ and the sender has input (L, m_0, m_1), where $L \in [n]$ represents an index of D. The goal of the two-message protocol is for the receiver to learn $(L, m_{D[L]})$, and nothing else. We will achieve this using laconic PSI (which can be achieved using laconic BPs). Suppose the receiver builds the set $A = \{iD_i0, iD_i1 \mid i \in [n]\}$, where D_i is the i-th element of D, and the sender builds the set $B = \{L0m_0, L1m_1\}$. As a first attempt, they run the laconic PSI protocol using BPs as described above. The intersection of these sets will be $A \cap B = \{LD_L0, LD_L1\} \cap \{L0m_0, L1m_1\}$, which in turn will be

$$A \cap B = \begin{cases} \{L00, L01\} \cap \{L0m_0, L1m_1\} & \text{if } D_L = 0. = \begin{cases} L00 & \text{if } m_0 = 0 \\ L01 & \text{if } m_0 = 1 \end{cases} \\ \{L10, L11\} \cap \{L0m_0, L1m_1\} & \text{if } D_L = 1. = \begin{cases} L10 & \text{if } m_1 = 0 \\ L11 & \text{if } m_1 = 1 \end{cases} \end{cases}.$$

From this, the receiver can take the third bit of the intersection output to be $m_{D[L]}$, achieving the desired output of laconic OT. But this does not directly translate to our BP protocol because the sender's set B has two elements. Our protocol needs to be run once per element of $|B|$, so running the protocol twice for $|B| = 2$ would no longer satisfy the requirements for laconic communication. Since we only need to do one additional intersection computation, this can be accommodated by allowing the sender's response message to be twice as large. One part of the message contains the information to find the intersection with $L0m_0$ and the other part of the message contains the information to find the intersection with $L1m_1$. As a result, the receiver's computational costs in the next step are doubled. Although inefficient, this does show that laconic OT can be realised with laconic BPs.

Two-party computation for BPs can be used to realize 2PC for PSI. Both laconic OT and laconic PSI can be viewed as special cases of a branching program. Recall that in the

setting of long sender input, where a sender has a large branching program, we have generic constructions from rate-1 OT which can be built from various assumptions. However, in the dual setting of long receiver input, we no longer have such a generic construction. Laconic OT *seems* to be a counterpart building block in the dual setting, but it does *not* give us laconic branching programs. Given the gap between the two settings, we ask the following question:

Can we achieve laconic branching programs from assumptions other than LWE?

This diversifies the set of assumptions from which laconic MPC can be realized. It also increases our understanding of how far each assumption allows us to expand the functionality, which helps in gaining insights into the theoretical limits of the assumptions themselves.

1.1 Our Results

In this work, we answer the above question in the affirmative. In the setting where the receiver holds a private large branching program, BP, and the sender holds a private short input x, we construct a two-round 2PC protocol allowing the receiver to learn x if and only if $\mathrm{BP}(x) = 1$, and nothing more. The communication only grows with $|x|$ and the *depth* of BP, and does not further depend on the size of BP. Furthermore, the sender's computation also only grows with $|x|$ and the depth of BP. Our construction is based on anonymous hash encryption schemes [BLSV18], which can in turn be based on CDH or LWE [DG17, BLSV18].

Sender Security. We achieve what we call *weak sender security* which says if $\mathrm{BP}(x) = 0$, then no information about x is leaked; else, there are no privacy guarantees for x. A stronger security requirement would be that in the latter case, the receiver should learn only $\mathrm{BP}(x)$, and no other information about x. Unfortunately, realizing strong sender security is too difficult in light of known barriers, because it generically implies a notion called *private laconic OT* [CDG+17, DGI+19]. Private laconic OT is laconic OT in which the index i chosen by the sender is also kept hidden from the receiver. The only existing construction of private laconic OT with polylogarithmic communication uses techniques from laconic secure function evaluation and is based on LWE [QWW18]. In particular, it is not known if private laconic OT can be realized using Diffie-Hellman assumptions.

Strong sender security allows one to achieve *laconic PSI cardinality*, a generalization of laconic PSI. In the PSI cardinality problem, the receiver learns only the size of the

set intersection and nothing about the intersection set itself. Strong sender security for a receiver with a large set S and a sender with a single element x would allow the receiver to learn whether or not $x \in S$, without learning anything extra about x. This immediately implies laconic PSI cardinality by having the sender send a second-round protocol message for each element in its set. We can get laconic PSI as an application of our results (and the other applications discussed below), but our results do not allow us to realize laconic PSI cardinality. Laconic PSI cardinality generically implies private laconic OT, establishing a barrier. The same barriers prevented [DKL+23] from building laconic PSI cardinality.

Applications. Our laconic branching program construction directly implies laconic OT and laconic PSI, as their functionalities can be represented as branching programs. Moreover, we can capture other functionalities not considered by previous work, such as private set union (PSU). A branching program for PSU can be obtained by making local changes to a branching program for PSI. (See Section 5.) This demonstrates the versatility of our approach, giving a unifying construction for all these functionalities. In contrast, the accumulator-based PSI constructions in [ABD+21, ALOS22, DKL+23] are crucially tied to the PSI setting, and do not seem to extend to the PSU setting. This is because the sender's message to the receiver only provides enough information to indicate which element (if any) in the receiver's set is also held by the sender. The receiver can not reconstruct the intersection element using the sender's message. On the other hand in the PSU setting, this is exactly what the receiver needs to do. If the sender's element is not in the receiver's set, the receiver needs to be able to recover the sender's element from the message.

Our techniques can be used in unbalanced PSI where the receiver holds a large set that can be represented as a branching program. For instance, a recent work by Garimella et al. [GRS22] introduced the notion of *structure-aware PSI* where one party's (potentially large) set Y is publicly known to have a certain structure. As long as the publicly known structure can be represented as a branching program, our techniques can be used to achieve a two-round PSI protocol where the communication only grows with the size of the smaller set $|X|$ and the *depth* of the branching program, and does not further depend on $|Y|$.

Chapter 2

Technical Overview

Our construction makes crucial use of the combination of garbled circuits [Yao86] with hash encryption (HE) [DG17, BLSV18]. Here, we provide a high-level overview of these tools. Following that, illustrations of our protocol demonstrate how they are used together.

In a garbled circuit scheme, the garbling of a circuit C produces the garbling $\tilde{\mathsf{C}}$ and a set of input wire label pairs $\{\mathsf{lb}_{i,b}\}_{i\in[n],b\in\{0,1\}}$, where n is the number of input wires for C. These outputs allow the computation of $\mathsf{Eval}(\tilde{\mathsf{C}}, \{\mathsf{lb}_{i,b}\}_{i\in[n],x[i]})$, which produces the output $y = \mathsf{C}(x)$. Let us consider what this means. Suppose, for example, the circuit C took inputs of length 4 bits. The set of garbled labels $\{\mathsf{lb}_{i,b}\}_{i\in[4],b\in\{0,1\}}$ can then be written as:

$$\{\mathsf{lb}_{i,b}\}_{i\in[4],b\in\{0,1\}} := \left\{ \begin{matrix} \mathsf{lb}_{1,0} & \mathsf{lb}_{2,0} & \mathsf{lb}_{3,0} & \mathsf{lb}_{4,0} \\ \mathsf{lb}_{1,1} & \mathsf{lb}_{2,1} & \mathsf{lb}_{3,1} & \mathsf{lb}_{4,1} \end{matrix} \right\}.$$

Each column of this label matrix represents an input wire, the pink row represents labels for a wire value of 0, and the green row represents labels for a wire value of 1. Consider the following evaluations of $\widetilde{\mathsf{C}}$:

$$\mathsf{Eval}\left(\tilde{\mathsf{C}}, \left\{ \mathsf{lb}_{1,0} \quad \mathsf{lb}_{2,0} \quad \mathsf{lb}_{3,0} \quad \mathsf{lb}_{4,0} \right\}\right) = \mathsf{C}(0000)$$
$$\mathsf{Eval}\left(\tilde{\mathsf{C}}, \left\{ \mathsf{lb}_{1,0} \quad \mathsf{lb}_{2,1} \quad \mathsf{lb}_{3,0} \quad \mathsf{lb}_{4,1} \right\}\right) = \mathsf{C}(0101)\,.$$

In the first example, evaluation on four 0 labels results in the value $\mathsf{C}(0000)$. One can obtain the value $\mathsf{C}(x)$ by evaluating $\tilde{\mathsf{C}}$ with the garbled labels associated with the bit values of x.

When used in context, a party will, of course, not be able to evaluate $\tilde{\mathsf{C}}$ so freely. Any

party wishing to evaluate $\tilde{\mathsf{C}}$ must, crucially, only have access to one label per wire. In other words, for all wires $i \in [n]$, no party can hold both $\mathsf{lb}_{i,0}$ and $\mathsf{lb}_{i,1}$. Security states that the garbled circuit together with the labels associated with some input x provides no more information than a copy of the circuit C and the output $y = \mathsf{C}(x)$. Note that security is destroyed if any party other than the garbler holds both $\mathsf{lb}_{i,0}$ and $\mathsf{lb}_{i,1}$ for any wire $i \in [n]$.

Hash encryption (HE), called batch encryption in [BLSV18], is a relaxation of Chameleon Encryption from [DG17]. The main difference between hash and chameleon encryption is that an HE scheme does not have a trapdoor algorithm for finding hash collisions. An HE scheme, parameterized by $n = n(\lambda)$, where λ is the security parameter, consists of a hash function $\mathsf{Hash}(\cdot) : \{0,1\}^n \to \{0,1\}^\lambda$ and associated HEnc and HDec functions.[1] One can encrypt n pairs of plaintexts $\mathbf{m} := \{m_{i,b}\}$ (for $i \in [n]$ and $b \in \{0,1\}$) with respect to a hash image $h := \mathsf{Hash}(z)$ to get $\mathsf{cth} \xleftarrow{\$} \mathsf{HEnc}(h, \mathbf{m})$. The ciphertext cth is such that given the hash pre-image z, one may recover half of the encrypted messages, in particular, the half associated with the bits of z: $\{m_{1,z_1}, \ldots, m_{n,z_n}\}$. Moreover, semantic security is maintained (even in the presence of z) for the other half of the encrypted plaintexts: $\{m_{i,1-z_i}\}_{i\in[n]}$. In other words, $2n$ messages are encrypted within cth, and z can only decrypt n plaintexts, leaving the remaining securely encrypted. Hash encryption can be constructed from the CDH or LWE assumptions [DG17, BLSV18].

[BLSV18] introduces *anonymous*[2] variants of these tools. In an anonymous garbled circuit scheme, the simulation algorithm has an additional property. If $y = \mathsf{C}(x)$ is uniformly random, then the output of $\mathsf{Sim}(\mathsf{C}, \mathsf{C}(x))$ is also uniformly random. In an anonymous hash encryption scheme, the ciphertext encrypting a uniformly random message is itself uniformly random. Anonymous garbled circuits and anonymous hash encryption can be used together to produce a chain of random circuit outputs.

Next, we provide two examples to illustrate our protocol for 2PC in which the receiver holds a BP and the sender holds a single element.

Depth 1 example. Consider a simple example where the receiver R has a depth-one BP on bits (see Def. 3 for branching programs) where the root node encodes index $i^* \in [n]$ (for $n \in \mathbb{N}$) and its left child encodes accept ($b_0 := \mathsf{accept}$) and its right child encodes reject ($b_1 := \mathsf{reject}$). This BP evaluates an input x by checking the bit value at index i^*. If $x[i^*] = 0$, then the value of the left child is output: $b_0 = \mathsf{accept}$. If $x[i^*] = 1$, then the value of the right child is output: $b_1 = \mathsf{reject}$. To start, suppose R only wants to learn if

[1] Hash and HEnc also take as input public parameters pp, which are omitted from this high-level discussion.

[2] called *blind* in [BLSV18].

$\mathsf{BP}(x) = 1$, where x is the sender's input. The receiver hashes $h := \mathsf{Hash}(\mathsf{pp}, (i^*, b_0, b_1))$, padding the input if necessary, and sends h to the sender, S. S has a circuit $\mathsf{F}[x]$, with their input x hardwired, such that on input (j, q_0, q_1), $\mathsf{F}[x]$ outputs $q_{x[j]}$. See Fig. 2.1 for circuit F. S garbles $\mathsf{F}[x]$ to get a garbled circuit $\widetilde{\mathsf{F}}$ and corresponding labels $\{\mathsf{lb}_{i,b}\}$. S uses the hash value, h, from R to compute $\mathsf{cth} \xleftarrow{\$} \mathsf{HEnc}(\mathsf{pp}, h, \{\mathsf{lb}_{i,b}\})$. Finally, S sends $(\widetilde{\mathsf{F}}, \mathsf{cth})$ to R. The receiver uses her hash pre-image value, $z := (i^*, \mathsf{accept}, \mathsf{reject})$, to recover $\{\mathsf{lb}_{i,z[i]}\}$: $\{\mathsf{lb}_{i,z[i]}\} \leftarrow \mathsf{HDec}(z, \mathsf{cth})$. This allows her to learn $\mathsf{F}[x](i^*, \mathsf{accept}, \mathsf{reject})$ from the garbled circuit by computing $\mathsf{Eval}(\widetilde{\mathsf{F}}, \{\mathsf{lb}_{i,z[i]}\})$. This value is output, indicating either accept ($\mathsf{BP}(x) = 1$) or reject ($\mathsf{BP}(x) = 0$).

Circuit $\mathsf{F}[x](j, q_0, q_1)$**:**	**Circuit** $\mathsf{V}[x, \{\mathsf{lb}_{i,b}\}](h'_0, h'_1, u)$**:**
HARDWIRED: Sender input x.	HARDWIRED: Sender input x and set of label pairs $\{\mathsf{lb}_{i,b}\}$.
OPERATION:	OPERATION:
If $x[j] = 0$ then return q_0	If $x[u] = 0$ then return $\mathsf{cth} \xleftarrow{\$} \mathsf{HEnc}(\mathsf{pp}, h'_0, \{\mathsf{lb}_{i,b}\})$
If $x[j] = 1$ then return q_1	If $x[u] = 1$ then return $\mathsf{cth} \xleftarrow{\$} \mathsf{HEnc}(\mathsf{pp}, h'_1, \{\mathsf{lb}_{i,b}\})$

Figure 2.1: **Circuits F and V for examples.**

Beyond depth 1. Next, consider the BP of depth 2 in Fig. 2.2 held by the receiver, R. Each internal node encodes an index: $\mathsf{root}, \mathsf{left}, \mathsf{right} \in [n]$. The four leaves have values with variables $(b_{00}, b_{01}, b_{10}, b_{11})$. For $i, j \in \{0, 1\}$, $b_{ij} \in \{\mathsf{accept}, \mathsf{reject}\}$. Suppose $x[\mathsf{root}] = 0$, where x is the sender's input, so the root-leaf path induced by $\mathsf{BP}(x)$ first goes left. If the sender, S, *somehow* knows the hash value $h_0 := \mathsf{Hash}(\mathsf{pp}, (\mathsf{left}, b_{00}, b_{01}))$, he can, as above, send a garbled circuit for $\mathsf{F}[x]$ and an HE ciphertext with respect to h_0 of the underlying labels, allowing R to evaluate $\mathsf{F}[x](\mathsf{left}, b_{00}, b_{01})$. But S does not know the value of h_0 nor whether the first step in the path is left or right (because the BP is hidden from S). Moreover, R cannot send both $h_0 := \mathsf{Hash}(\mathsf{pp}, (\mathsf{left}, b_{00}, b_{01}))$ and $h_1 := \mathsf{Hash}(\mathsf{pp}, (\mathsf{right}, b_{10}, b_{11}))$. The first reason is that there would be a size blow-up since the communication cost would grow with the size, and not the depth, of the BP. Secondly, R would learn more information than necessary. Since S does not know *a priori* whether the induced path travels left or right on the BP, he has to encrypt the labels under both h_0 and h_1. But encrypting the labels $\{\mathsf{lb}_{i,b}\}$ under both h_0 and h_1 allows the receiver to recover two labels for an index on which $(\mathsf{left}, b_{00}, b_{01})$ and $(\mathsf{right}, b_{10}, b_{11})$ differ, destroying garbled-circuit security.

FIXING SIZE BLOW-UP VIA DEFERRED ENCRYPTION. We fix the above issue via deferred encryption techniques [DG17, BLSV18, GHMR18, ABD+21], allowing the sender to defer

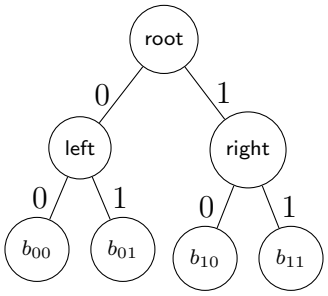


Figure 2.2: **Depth 2 BP example.**

the HE encryptions of $\{\mathsf{lb}_{i,b}\}$ labels to the receiver herself at decryption time! To enable this technique, the receiver further hashes (h_0, h_1) such that during decryption, the receiver, through the evaluation of a garbled circuit, will obtain an HE encryption of $\{\mathsf{lb}_{i,b}\}$ with respect to $h_{x[\mathsf{root}]}$, where $\{\mathsf{lb}_{i,b}\}$ and (h_0, h_1) are as above. In other words, the receiver obtains $\mathsf{HEnc}(\mathsf{pp}, h_{x[\mathsf{root}]}, \{\mathsf{lb}_{i,b}\})$. To do this, we have to explain how the receiver further hashes h_0 and h_1, and how she can later perform deferred encryption. First, the receiver R computes the hash value $\mathsf{hr} := \mathsf{Hash}(\mathsf{pp}, (h_0, h_1, \mathsf{root}))$, and sends hr to S. Next the sender S, with input x, garbles $\mathsf{F}[x]$ (as in Fig. 2.1) to get $(\widetilde{\mathsf{F}}, \{\mathsf{lb}_{i,b}\})$ as above. Then, he forms a circuit $\mathsf{V}[x, \{\mathsf{lb}_{i,b}\}]$ with x and $\{\mathsf{lb}_{i,b}\}$ hardwired, which on input (h'_0, h'_1, u) outputs $\mathsf{HEnc}(h'_{x[u]}, \{\mathsf{lb}_{i,b}\})$. See Fig. 2.1 for circuit V. The sender garbles $\mathsf{V}[x, \{\mathsf{lb}_{i,b}\}]$ to get $(\widetilde{\mathsf{V}}, \{\mathsf{lb}'_{i,b}\})$. If R is given $\widetilde{\mathsf{V}}$ and the labels $\{\mathsf{lb}'_{i,z'[i]}\}$, where $z' := (h_0, h_1, \mathsf{root})$, she can evaluate $\widetilde{\mathsf{V}}$ on these labels, which will in turn release an HE encryption of labels $\{\mathsf{lb}_{i,b}\}$ under $h_{x[\mathsf{root}]}$, as desired. To ensure R only gets the $\{\mathsf{lb}'_{i,z'[i]}\}$ labels, S encrypts the $\{\mathsf{lb}'_{i,b}\}$ labels under hr, and sends the resulting HE ciphertext cth', as well as $\widetilde{\mathsf{F}}$ and $\widetilde{\mathsf{V}}$ to R. From cth' and $z' := (h_0, h_1, \mathsf{root})$, R can only recover the labels $\{\mathsf{lb}'_{i,z'[i]}\}$, as desired.

Receiver decryption. The receiver will evaluate $\widetilde{\mathsf{V}}$ on the decrypted $\{\mathsf{lb}'_{i,z'[i]}\}$ labels, releasing cth: an HE encryption under $h_{x[\mathsf{root}]}$ of the label pairs $\{\mathsf{lb}_{i,b}\}$. The receiver does not know whether cth is encrypted under h_0 or h_1 (since she does not know the value $x[\mathsf{root}]$), so she tries to decrypt with respect to the pre-images of both hash values and checks which one (if any) is valid. However, this results in the following security issue: an HE scheme is not guaranteed to hide the underlying hash value with respect to which an HE ciphertext was made. For example, consider a semantically secure HE scheme where $\mathsf{HEnc}(\mathsf{pp}, h, \{m_{i,b}\})$ appends h to the ciphertext. Employing such an HE scheme in the above construction signals to the receiver if cth' was encrypted under h_0 or h_1, which reveals the bit value of $x[\mathsf{root}]$. This breaks sender security when $\mathsf{BP}(x) = 0$. Moreover,

even if the HE encryption scheme is anonymous in the sense of hiding h, decrypting an h_b-formed HE ciphertext under the pre-image of h_{1-b} may result in $\perp$, or in junk labels that do not work on $\widetilde{\mathsf{F}}$, which again causes the breaking of sender security. To resolve this issue, we use the same technique as in [ABD+21] of using anonymous hash encryption with anonymous garbled circuits.

Signalling the correct output of F. In the above examples, $\mathsf{F}[x]$ outputs either accept or reject, indicating if $\mathsf{BP}(x)$ equals 1 or 0, respectively. But in the desired functionality, $\mathsf{F}[x]$ outputs x if $\mathsf{BP}(x) = 1$. We cannot simply modify $\mathsf{F}[x]$ to output x if $q_{x[j]} = \mathsf{accept}$ since in that case if the receiver evaluates $\widetilde{\mathsf{F}}$ on junk labels she will not be able to tell the difference between the junk output and x. Similar to [ABD+21], we address this problem by having S include a signal value in their message to R and hardcoded in the circuits. Then we can modify $\mathsf{F}[x]$ to output x and the signal value. The receiver compares this output signal value to the true value contained in the sender's message. If they are equal, R knows that the output x is equal to the sender's input x.

Handling unbalanced branching programs. The above depth two discussion can be naturally extended to the balanced BP setting, wherein we have a full binary tree of depth d. When the BP is unbalanced, like our BPs for PSI and PSU, the above approach fails because the sender does not know *a priori* which branches terminate early. We solve this issue via the following technique. In the above examples, the receiver's evaluation of a path terminates once the maximum depth of the BP is reached. Internal nodes are evaluated using the V circuit and leaf nodes are evaluated using the F circuit. We modify V to detect from its inputs if the current path has ended before the maximum depth is reached, and if so, it halts and outputs the appropriate values. In halting mode, the circuit V will release its hardwired input x, assuming the halt is an accept. Executing the above blueprint requires striking a delicate balance to have both correctness and security.

Comparison with [DGGM19]. The work of Döttling, Garg, Goyal, and Malavolta [DGGM19] builds laconic conditional disclosure of secrets (CDS) in which a sender $\mathsf{S}(x, m)$ holding an NP instance x and a message m, and a receiver holding x and a potential witness w for x. If $R(x, w) = 1$, where R is the corresponding relation R, the receiver learns m; otherwise, the receiver learns no information about m. They show how to build two-round laconic CDS protocols with polylogarithmic communication and polylogarithmic sender's computation from CDH or LWE.

Note that the CDS setting is incomparable to ours. The closest resemblance is to think of the BP input x as the NP instance, and of the BP as the NP witness w—but then under

CDS the input x is not kept hidden from the receiver. In particular, it is not even clear whether laconic CDS implies laconic PSI.

Comparison with [ABD+21]. At a high level, the garbled circuit-based laconic PSI construction of [ABD+21] is an ad hoc and specific instantiation of our general methodology. In particular, for a receiver with $m = 2^k$ elements (for $k := \mathsf{polylog}(\lambda)$), the construction of [ABD+21] builds a full binary tree of depth k, with the m elements stored in the leaves, Merkle hashed up the tree in a specific way. Namely, the pre-image of each node's hash value is comprised of its two children's hashes as well as some additional encoded information about its sub-tree, enabling an evaluator, with an input x, to make a deterministic left-or-right downward choice at each intermediate node. This is a very specific BP instantiation of PSI, where the intermediate BP nodes, instead of running index predicates (e.g., travelling left or right if the i-th bit is 0 or 1), they run full-input predicates $\Phi : x \mapsto \{0, 1\}$, where Φ is defined based on the left sub-tree of the node. Our approach, on the other hand, handles branching programs for index predicates, and we subsume the results of [ABD+21] as a special case. In particular, we show how to design simple index-predicate BPs for PSI, PSU, and wildcard matching, the latter two problems are not achieved by [ABD+21].

In summary, our construction generalizes and simplifies the approach of [ABD+21], getting much more mileage out of the garbled-circuit based approach. For example, [ABD+21] builds a secure protocol for a specific PSI-based BP which is in fact a decision tree: namely, the in-degree of all internal nodes is one. On the other hand, we generalize this concept to handle all decision trees and even the broader class of branching programs, in which the in-degree of intermediate nodes can be greater than one. Moreover, we introduce some new techniques (e.g., for handling unbalanced BPs) that may be of independent interest.

Chapter 3

Preliminaries

The acronym PPT denotes "probabilistic polynomial time". Throughout this work, λ denotes the security parameter. $\mathsf{negl}(\lambda)$ denotes a negligible function in λ, that is, a function that vanishes faster than any inverse polynomial in λ.

For $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, \ldots, n\}$. For a bit string x, $\bar{x}$ denotes the complement string, namely the string x with all bit values flipped. If $x \in \{0,1\}^n$ then the bits of x can be indexed as $x[i] := x_i$ for $i \in [n]$, where $x = x_1 \ldots x_n$ (note that indexing begins at 1, not 0). $x := y$ is used to denote the assignment of variable x to the value y. If $\mathcal{A}$ is a deterministic algorithm, $y \leftarrow \mathcal{A}(x)$ denotes the assignment of the output of $\mathcal{A}(x)$ to variable y. If $\mathcal{A}$ is randomized, $y \xleftarrow{\$} \mathcal{A}(x)$ is used. If S is a (finite) set, $x \xleftarrow{\$} S$ denotes the experiment of sampling uniformly at random an element x from S. If D is a distribution over S, $x \xleftarrow{\$} D$ denotes the element x sampled from S according to D. If D_0, D_1 are distributions, we say that D_0 is statistically (resp. computationally) indistinguishable from D_1, denoted by $D_0 \approx_s D_1$ (resp. $D_0 \stackrel{c}{\equiv} D_1$), if no unbounded (resp. PPT) adversary can distinguish between the distributions except with probability at most $\mathsf{negl}(\lambda)$.

If Π is a two-round two-party protocol, then $(\mathsf{m}_1, \mathsf{m}_2) \leftarrow \mathsf{tr}^{\Pi}(x_0, x_1, \lambda)$ denotes the protocol transcript, where x_i is party P_i's input for $i \in \{0,1\}$. For $i \in \{0,1\}$, $(x_i, r_i, \mathsf{m}_1, \mathsf{m}_2) \leftarrow \mathsf{view}_i^{\Pi}(x_0, x_1, \lambda)$ denotes P_i's "view" of the execution of Π, consisting of their input, random coins, and the protocol transcript.

Definition 1 (Computational Diffie-Hellman (CDH))**.** *Let $\mathcal{G}(1^\lambda)$ be an algorithm that outputs $(\mathbb{G}, p, g)$ where $\mathbb{G}$ is a group of prime order p and g is a generator of the group.*

The CDH assumption holds for generator $\mathcal{G}$ if for all PPT adversaries $\mathcal{A}$

$$\Pr\left[g^{a_1 a_2} \leftarrow \mathcal{A}(\mathbb{G}, p, g, g^{a_1}, g^{a_2}) : \begin{array}{c}(\mathbb{G}, p, g) \leftarrow \mathcal{G}(\lambda) \\ a_1, a_2 \leftarrow\!\!\$\ \mathbb{Z}_p\end{array}\right] \leq \mathsf{negl}(\lambda).$$

Definition 2 (Learning with Errors)**.** *Let $q, k \in \mathbb{N}$ where $k \in \mathsf{poly}(\lambda)$, $\mathbf{A} \in \mathbb{Z}_q^{k \times n}$ and $\beta \in \mathbb{R}$. For any $n = \mathsf{poly}(k \log q)$, the LWE assumption holds if for every PPT algorithm $\mathcal{A}$ we have*

$$\left|\Pr\left[1 \leftarrow \mathcal{A}(\mathbf{A}, \mathbf{sA} + \mathbf{e})\right] - \Pr\left[1 \leftarrow \mathcal{A}(\mathbf{A}, \mathbf{y})\right]\right| \leq \mathsf{negl}(\lambda)\,,$$

for $\mathbf{s} \xleftarrow{\$} \{0,1\}^k$, $\mathbf{e} \xleftarrow{\$} D_{\mathbb{Z}^n, \beta}$ and $\mathbf{y} \xleftarrow{\$} \{0,1\}^n$, where $D_{\mathbb{Z}^n, \beta}$ is some error distribution.

The following definitions related to branching programs are modified from [IP07].

Definition 3 (Branching Program (BP))**.** *A (deterministic) branching program over the input domain $\{0,1\}^\lambda$ and output domain $\{0,1\}$ is defined by a tuple (V, E, T, Val) where:*

- *$G := (V, E)$ is a rooted, directed, acyclic graph of depth d.*
- *Two types of nodes partition V:*
  - *Interior nodes: Have outdegree 2.*[1] *The root node, denoted $v_1^{(0)}$, has indegree 0.*
  - *Terminal/leaf nodes: Have outdegree 0. T denotes the set of terminal nodes. Leaf nodes are labeled as $T = \{u_1, \ldots, u_{|T|}\}$. Each $u_i \in T$ encodes a value in $\{0,1\}$.*
- *Each node in V encodes a value in $[\lambda]$. These values are stored in the array Val such that for all $v \in V \setminus T$, $\mathsf{Val}[v] = i$ for some $i \in [\lambda]$ and for all $u \in T$, $\mathsf{Val}[u] \in \{0,1\}$.*
- *The elements of the edge set E are formatted as an ordered tuple (v, v', b) indicating a directed edge from $v \in V$ to $v' \in V$ with label $b \in \{0,1\}$. For every interior node v with left child v'_0 and right child v'_1, the edges $(v, v'_0, 0)$ and $(v, v'_1, 1)$ are in E.*

BP Evaluation. The output of a branching program is defined by the function $\mathsf{BP} : \{0,1\}^\lambda \to \{0,1\}$, which on input $x \in \{0,1\}^\lambda$ outputs a bit. Evaluation of BP (see Fig. 3.1, right, and relevant function definitions below) follows the unique path in G induced by x from the root $v_1^{(0)}$ to a leaf node $u \in T$. The output of BP is the value encoded in u, $\mathsf{Val}[u]$.

[1] We assume no nodes have outdegree 1 since such nodes can be removed from the BP w.l.o.g.

$\mathsf{Eval}_{\mathsf{int}}(v, x)$:	$\mathsf{BP}(x)$:
$i \leftarrow \mathsf{Val}[v]$	$v \leftarrow v_1^{(0)}$
If $x[i] = 0$ then return $\Gamma(v, 0)$	While $v \notin T$ do
Else return $\Gamma(v, 1)$	$\quad v \leftarrow \mathsf{Eval}_{\mathsf{int}}(v, x)$
	$y \leftarrow \mathsf{Eval}_{\mathsf{leaf}}(v)$
	Return y

Figure 3.1: **Interior node evaluation function $\mathsf{Eval}_{\mathsf{int}}$ and BP evaluation function BP.**

- $\Gamma : V \setminus T \times \{0,1\} \to V$ takes as input an internal node v and a bit b and outputs v's left child if $b = 0$ and v's right child if $b = 1$.
- $\mathsf{Eval}_{\mathsf{int}} : V \setminus T \times \{0,1\}^\lambda \to V$ takes as input an interior node v and a string of length λ and outputs either v's left or right child ($\Gamma(v, 0)$ or $\Gamma(v, 1)$, respectively). See Figure 3.1, left.
- $\mathsf{Eval}_{\mathsf{leaf}} : T \to \{0,1\}$ takes as input a terminal node $u \in T$ and outputs the value $\mathsf{Val}[u]$.

Definition 4 (Layered BP)**.** *A BP of depth d is layered if the node set V can be partitioned into $d+1$ disjoint levels $V = \bigcup_{i=0}^{d} V^{(i)}$, such that $V^{(0)} = \{v_1^{(0)}\}$, $V^{(d)} \subseteq T$, and for every edge $e = (u, v, b)$ (where b is the edge label) we have $u \in V^{(i)}$, $v \in V^{(i+1)}$ for some level $i \in \{0, \ldots, d-1\}$. We refer to $V^{(i)}$ as the i-th level of the BP, or the level at depth i. Nodes on level i are labelled $V^{(i)} = \{v_1^{(i)}, \ldots, v_{|V^{(i)}|}^{(i)}\}$.*

We require that all branching programs in this work are layered.

3.1 Hash Encryption + Garbled Circuits

Our construction uses hash encryption schemes with garbled circuits. The following definitions are taken directly from [ABD+21].

Definition 5 (Hash Encryption [DG17, BLSV18][2])**.** *A hash encryption scheme* $\mathsf{HE} = (\mathsf{HGen}, \mathsf{Hash}, \mathsf{HEnc}, \mathsf{HDec})$ *is defined as follows.*

[2]Hash encryption is called *batch encryption* in [BLSV18].

- $\mathsf{HGen}(1^\lambda, n)$: *Takes as input a security parameter* 1^λ *and an input size* n*, and outputs a hash key* pp*.*

- $\mathsf{Hash}(\mathsf{pp}, z)$: *Takes as input a hash key* pp *and* $z \in \{0,1\}^n$*, and deterministically outputs* $h \in \{0,1\}^\lambda$*.*

- $\mathsf{HEnc}(\mathsf{pp}, h, \{m_{i,b}\}_{i\in[n],b\in\{0,1\}}; \{r_{i,b}\})$: *Takes as input a hash key* pp*, a hash output* h*, messages* $\{m_{i,b}\}$ *and randomness* $\{r_{i,b}\}$*, and outputs* $\{\mathsf{cth}_{i,b}\}_{i\in[n],b\in\{0,1\}}$*. We write it concisely as* $\{\mathsf{cth}_{i,b}\}$*. Each ciphertext* $\mathsf{cth}_{i,b}$ *is computed as* $\mathsf{cth}_{i,b} = \mathsf{HEnc}(\mathsf{pp}, h, m_{i,b}, (i,b); r_{i,b})$*, where we have overloaded the* HEnc *notation.*

- $\mathsf{HDec}(z, \{\mathsf{cth}_{i,b}\})$: *Takes as input a hash input* z *and* $\{\mathsf{cth}_{i,b}\}$ *and deterministically outputs* n *messages* $(m_1, \ldots, m_n)$*.* Correctness *requires that* $(m_1, \ldots, m_n) = (m_{1,z[1]}, \ldots, m_{n,z[n]})$*, where* z *was the hash pre-image used to encrypt* $\{\mathsf{cth}_{i,b}\}$*.*

A hash encryption scheme must satisfy the following semantic security property. Anonymous semantic security is additionally required for anonymous *HE schemes.*

- ***Semantic Security:*** *Given* $z \in \{0,1\}^n$*, no adversary can distinguish between encryptions of messages made to indices* $(i, \bar{z}_i)$*. For any PPT* $\mathcal{A}$*, sampling* $\mathsf{pp} \xleftarrow{\$} \mathsf{HGen}(1^\lambda, n)$*, if* $(z, \{m_{i,b}\}, \{m'_{i,b}\}) \xleftarrow{\$} \mathcal{A}(\mathsf{pp})$ *and if* $m_{i,z[i]} = m'_{i,z[i]}$ *for all* $i \in [n]$*, then* $\mathcal{A}$ *cannot distinguish between* $\mathsf{HEnc}(\mathsf{pp}, h, \{m_{i,b}\})$ *and* $\mathsf{HEnc}(\mathsf{pp}, h, \{m'_{i,b}\})$*, where* $h \leftarrow \mathsf{Hash}(\mathsf{pp}, z)$*.*

- ***Anonymous Semantic Security:*** *For a random* $\{m_{i,b}\}$ *with equal rows (i.e.,* $\forall i \in [n], m_{i,0} = m_{i,1}$*), the output of* $\mathsf{HEnc}(\mathsf{pp}, h, \{m_{i,b}\})$ *is pseudorandom even in the presence of the hash pre-image. Formally, for any* $z \in \{0,1\}^n$*, sampling* $\mathsf{pp} \xleftarrow{\$} \mathsf{HGen}(1^\lambda, n)$*,* $h \leftarrow \mathsf{Hash}(\mathsf{pp}, z)$*, and sampling* $\{m_{i,b}\}$ *uniformly at random with the same rows, then* $\boldsymbol{v} := (\mathsf{pp}, z, \mathsf{HEnc}(\mathsf{pp}, h, \{m_{i,b}\}))$ *is indistinguishable from another tuple in which we replace the hash-encryption component of* $\boldsymbol{v}$ *with a random string.*

The following results are from [BLSV18, GGH19].

Lemma 1. *Assuming CDH or LWE, there exist anonymous hash encryption schemes, where* $n = 3\lambda$ *(i.e.,* $\mathsf{Hash}(\mathsf{pp}, \cdot)\colon \{0,1\}^{3\lambda} \mapsto \{0,1\}^\lambda$*).*[3] *Moreover, the hash function* Hash

[3]The CDH construction of [BLSV18] satisfies a weaker notion of anonymity, in which only some part of the ciphertext is pseudorandom. This weaker notion of anonymity is sufficient for our construction, but requires cumbersome notational tweaks. So for ease of presentation, we keep the notion as is.

satisfies robustness *in the following sense: for any input distribution on z which samples at least 2λ bits of z uniformly at random, $(\mathsf{pp}, \mathsf{Hash}(\mathsf{pp}, z))$ and (pp, u) are statistically close, where $\mathsf{pp} \xleftarrow{\$} \mathsf{HGen}(1^\lambda, 3\lambda)$ and $u \xleftarrow{\$} \{0,1\}^\lambda$.*

We also review garbled circuits and the anonymous property, as defined in [BLSV18].

Definition 6 (Garbled Circuits)**.** *A garbling scheme for a class of circuits $\mathcal{C} := \{\mathsf{C}\colon \{0,1\}^n \mapsto \{0,1\}^m\}$ consists of $(\mathsf{Garb}, \mathsf{Eval}, \mathsf{Sim})$ satisfying the following.*

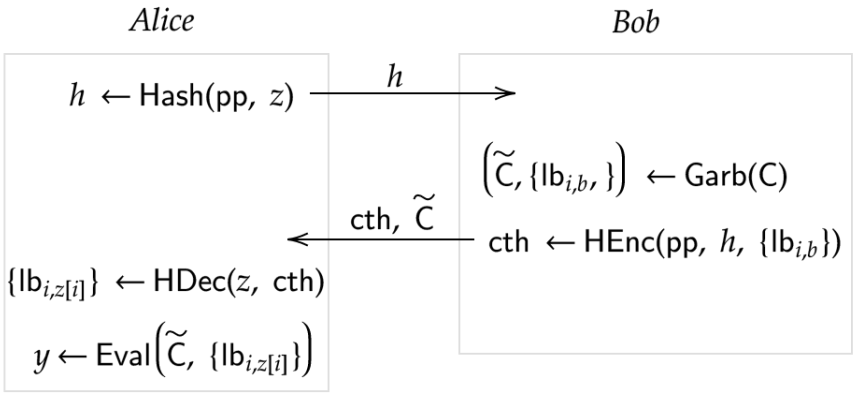
- ***Correctness**: For all $\mathsf{C} \in \mathcal{C}$ and $x \in \{0,1\}^n$, $\Pr[\mathsf{Eval}(\widetilde{\mathsf{C}}, \{\mathsf{lb}_{i,x[i]}\}) = \mathsf{C}(x)] = 1$, where $(\widetilde{\mathsf{C}}, \{\mathsf{lb}_{i,b}\}) \xleftarrow{\$} \mathsf{Garb}(1^\lambda, \mathsf{C})$.*
- ***Simulation Security**: For any $\mathsf{C} \in \mathcal{C}$ and $x \in \{0,1\}^n$: $(\widetilde{\mathsf{C}}, \{\mathsf{lb}_{i,x[i]}\}) \stackrel{c}{\equiv} \mathsf{Sim}(1^\lambda, \mathsf{C}, y)$, where $(\widetilde{\mathsf{C}}, \{\mathsf{lb}_{i,b}\}) \xleftarrow{\$} \mathsf{Garb}(1^\lambda, \mathsf{C})$ and $y \leftarrow \mathsf{C}(x)$.*
- ***Anonymous Security**[4] [BLSV18]: For any $\mathsf{C} \in \mathcal{C}$ and random $y \xleftarrow{\$} \{0,1\}^m$, the output of $\mathsf{Sim}(1^\lambda, \mathsf{C}, y)$ is pseudorandom.*

Lemma 2 ([BLSV18])**.** *Anonymous garbled circuits can be built from one-way functions.*

Notation for Hash Encryption. We assume $\mathsf{Hash}(\mathsf{pp}, \cdot)\colon \{0,1\}^n \to \{0,1\}^\lambda$, where $n = 3\lambda$. For $i \in [n]$ and $b \in \{0,1\}$, $\{\mathsf{lb}_{i,b}\}$ denotes a sequence of n label pairs. For all $i \in [n]$, all $b \in \{0,1\}$, and $\boldsymbol{r} := \{r_{i,b}\}$, $\mathsf{HEnc}(\mathsf{pp}, h, \{\mathsf{lb}_{i,b}\}; \boldsymbol{r})$ denotes a set of ciphertexts $\{\mathsf{cth}_{i,b}\}$, where the (i,b)-th ciphertext is defined as $\mathsf{cth}_{i,b} \leftarrow \mathsf{HEnc}(\mathsf{pp}, h, \mathsf{lb}_{i,b}, (i,b); r_{i,b})$. We overload notation as follows. For all $i \in [n]$, $\{\mathsf{lb}_i\}$ denotes a sequence of 3λ labels. For all $i \in [n]$ and $\boldsymbol{r} := \{r_{i,b}\}$, $\mathsf{HEnc}(\mathsf{pp}, h, \{\mathsf{lb}_i\}; \boldsymbol{r})$ denotes a hash encryption where both plaintext rows are $\{\mathsf{lb}_i\}$; namely, ciphertexts $\{\mathsf{cth}_{i,b}\}$, where for all $i \in [n]$, $\mathsf{cth}_{i,b} \leftarrow \mathsf{HEnc}(\mathsf{pp}, h, m_{i,b}, (i,b); r_{i,b})$ and $m_{i,0} = m_{i,1} = \mathsf{lb}_i$, for all i.

Previously, we informally discussed the power couple made by garbling and hash encryption, now with the definitions of these tools in hand we revisit the discussion more formally. Consider the following example to illustrate how garbled circuits will be used with hash encryption. Let $(\mathsf{Garb}, \mathsf{Eval}, \mathsf{Sim})$ be a garbling scheme for the class of circuits $\mathcal{C} := \{\mathsf{C} : \{0,1\}^n \mapsto \{0,1\}^m\}$. Let Alice generate $\mathsf{pp} \xleftarrow{\$} \mathsf{HGen}(1^\lambda, n)$ and $z \in \{0,1\}^n$. Let Bob hold the circuit $\mathsf{C} \in \mathcal{C}$ and get pp from Alice. The two parties can then engage in the following:

[4] called *blind* garbled circuits in [BLSV18].



At the end of this interaction, Alice's evaluation of the garbled circuit allows her to learn $y = \mathsf{C}(z)$. Her message to Bob commits her to *only* learning this one output of C. Security of the HE scheme ensures the labels encrypted in cth *not* corresponding to the bits of z remain secure. Without these labels, Alice cannot evaluate $\tilde{\mathsf{C}}$ on any other input, which means garbled circuit security is maintained. Unlike standard chosen plaintext attack (CPA)-secure PKC schemes, hash encryption only allows Alice to decrypt *half* of the encrypted plaintexts. Indeed, if Alice could decrypt the entirety of cth above, she would obtain garbled labels $\{\mathsf{lb}_{i,b}\}$ for all $i \in [n], b \in \{0,1\}$. This would allow her to learn $\mathsf{C}(x)$ for all $x \in \{0,1\}^n$, completely obliterating Bob's garbled circuit security! Hash encryption allows Alice to keep z, her desired input to C, hidden from Bob while committing to only learning $\mathsf{C}(z)$ from the garbled information she obtains from Bob.

Chapter 4

Semi-Honest Laconic 2PC with Branching Programs

Definition 7 (BP-2PC Functionality)**.** *We define the evaluation of a branching program in the two-party communication setting (*BP-2PC*) to be a two-round protocol between a receiver, R, and a sender, S, such that:*

- *R holds a branching program* BP *with evaluation function* $\mathsf{BP} : \{0,1\}^\lambda \to \{0,1\}$ *and S holds a string* $\mathsf{id} \in \{0,1\}^\lambda$*. In the first round of the protocol, R sends the message* m_1 *to S. In the second round S sends* m_2 *to R.*
- ***Correctness:*** *If* $\mathsf{BP}(\mathsf{id}) = 1$*, then R outputs* id*. Otherwise, R outputs $\perp$.*
- ***Computational (resp., statistical) receiver security:*** BP-2PC *achieves receiver security if for all* $\mathsf{id} \in \{0,1\}^\lambda$*, and all pairs of branching programs* $\mathrm{BP}_0, \mathrm{BP}_1$ *we have that* $\mathsf{view}_S^{\mathsf{BP\text{-}2PC}}(\mathrm{BP}_0, \mathsf{id}, \lambda) \approx \mathsf{view}_S^{\mathsf{BP\text{-}2PC}}(\mathrm{BP}_1, \mathsf{id}, \lambda)$*. If the distributions are computationally (resp., statistically) indistinguishable then we have computational (resp., statistical) security.*
- ***Computational (resp., statistical) sender security:*** BP-2PC *achieves sender security if for all branching programs* BP*, and all pairs* $\mathsf{id}_0, \mathsf{id}_1 \in \{0,1\}^\lambda$ *with* $\mathsf{BP}(\mathsf{id}_0) = 0 = \mathsf{BP}(\mathsf{id}_1)$*, we have that* $\mathsf{view}_R^{\mathsf{BP\text{-}2PC}}(\mathrm{BP}, \mathsf{id}_0, \lambda) \approx \mathsf{view}_R^{\mathsf{BP\text{-}2PC}}(\mathrm{BP}, \mathsf{id}_1, \lambda)$*. If the distributions are computationally (resp., statistically) indistinguishable then we have computational (resp., statistical) security.*

- ***Security against outsiders:*** BP-2PC *is secure against outsiders if for all outside parties* $P \notin \{S, R\}$ *and all pairs of sender/receiver inputs* $(\mathrm{BP}_0, \mathsf{id}_0)$ *and* $(\mathrm{BP}_1, \mathsf{id}_1)$*, we have that* $\mathsf{tr}^{\mathsf{BP\text{-}2PC}}(\lambda, \mathrm{BP}_0, \mathsf{id}_0) \approx \mathsf{tr}^{\mathsf{BP\text{-}2PC}}(\lambda, \mathrm{BP}_1, \mathsf{id}_1)$.

4.1 The BP-2PC Construction

In this section, we give a construction for a BP-2PC protocol, inspired by laconic OT techniques [CDG+17, ABD+21]. Construction 1 uses hash encryption and garbling schemes. A high-level overview follows.

1. The receiver party R hashes their branching program in a specific way from the leaf level up to the root. R then sends the message $\mathsf{m}_1 = \mathsf{h}_{\mathsf{root}}$ to the sender, where $\mathsf{h}_{\mathsf{root}}$ is the hash value of the root node of the hashed BP.

2. The sender party S gets the message $\mathsf{m}_1 = \mathsf{h}_{\mathsf{root}}$ and garbles one circuit for each level of a hash tree with $\lambda + 1$ levels, $\lambda + 1$ being the number of levels required for every bit of the sender's length λ input to be checked. $\mathsf{d}_{\mathsf{m}} := \lambda$ denotes this maximum depth. S starts with the leaf level and garbles circuit F (Fig. 4.1). F takes as input a leaf node value and two random strings. If the leaf node value is 1, F outputs the hardcoded sender element id and a random, fixed, signal string r. Otherwise, F outputs two random strings (id', r'). Then for every level from the leaf parents to the root, S garbles the circuit V (also in Fig. 4.1). Each V garbled by the sender has the labels of the previously generated garbled circuit hardcoded. After garbling, S computes a hash encryption of the root-level garbled circuit labels using the hash image $\mathsf{h}_{\mathsf{root}}$. Finally, S sends $\mathsf{m}_2 := (\widetilde{\mathsf{C}}_{\mathsf{d}_{\mathsf{m}}}, \ldots, \widetilde{\mathsf{C}}_0, \{\mathsf{cth}^{(0)}_{i,b}\}, r)$ to R, where $\widetilde{\mathsf{C}}_w$ is the garbled circuit associated with level w, $\{\mathsf{cth}^{(0)}_{i,b}\}$ is the encryption of the labels for $\widetilde{\mathsf{C}}_0$, and r is the signal value.

3. For all root-to-leaf paths through the BP, R runs DecPath (Fig. 4.2) on m_2 searching for the path that will decrypt to a signal value equal to r from m_2. On input a path pth and m_2, DecPath outputs a pair $(\mathsf{id}_{\mathsf{pth}}, r_{\mathsf{pth}})$ to R. If $r_{\mathsf{pth}} = r$, then R takes $\mathsf{id}_{\mathsf{pth}}$ to be S's element.

Construction 1 (BP-2PC)**.** *We require the following ingredients for the two-round, two-party communication BP construction.*

1. *An anonymous and robust hash encryption scheme* $\mathsf{HE} = (\mathsf{HGen}, \mathsf{Hash}, \mathsf{HEnc}, \mathsf{HDec})$*, where* $\mathsf{Hash}(\mathsf{pp}, \cdot)\colon \{0,1\}^{3\lambda} \mapsto \{0,1\}^{\lambda}$.

2. *An anonymous garbling scheme* $\mathsf{GS} = (\mathsf{Garb}, \mathsf{Eval}, \mathsf{Sim})$.

3. Circuits F *and* V *defined in Figure 4.1. Procedure* DecPath*, defined in Figure 4.2.*

We assume the receiver holds a—potentially unbalanced—branching program BP of depth $d \leq \lambda + 1$ *as defined in Def. 3. The sender has a single element* $\mathsf{id} \in \{0,1\}^\lambda$*.* $\mathsf{BP\text{-}2PC} := (\mathsf{GenCRS}, \mathsf{R}_1, \mathsf{S}, \mathsf{R}_2)$ *is a triple of algorithms built as follows.*

$\mathsf{GenCRS}(1^\lambda)$**:** *Return* $\mathsf{crs} \xleftarrow{\$} \mathsf{HGen}(1^\lambda, 3\lambda)$*.*

$\mathsf{R}_1(\mathsf{crs}, \mathsf{BP})$**:** *Recall from Def. 3 that BP has terminal node set* $T = \{u_1, \ldots, u_{|T|}\}$*, nodes in level* $0 \leq w \leq d$ *are labelled from leftmost to rightmost:* $V^{(w)} = \{v_1^{(w)}, \ldots, v_{|V^{(w)}|}^{(w)}\}$*.*

- *Parse* $\mathsf{crs} := \mathsf{pp}$*. The receiver creates a hashed version of BP, beginning at the leaf level: For* $j \in [|T|]$*, sample* $x_j, x'_j \xleftarrow{\$} \{0,1\}^\lambda$ *and compute* $h_j^{(d)} \leftarrow \mathsf{Hash}(\mathsf{pp}, (\mathsf{Val}[u_j]^{\times\lambda}, x_j, x'_j))$*.* $\mathsf{Val}[u_j]^{\times\lambda}$ *indicates that* $\mathsf{Val}[u_j]$ *is copied* λ *times to obtain either the all zeros or all ones string of length* λ*.*

  The remaining levels are hashed from level $d-1$ *up to* 0 *(the root): (An example is given in Appendix A, figures A.1 and A.2.)*

  - *For* w *from* $d-1$ *to* 0*,* $|V^{(w)}|$ *nodes are added to level* w*. The hash value of each node is the hash of the concatenation of its left child, right child, and the index encoded in the current node. Formally: For* $j \in [|V^{(w)}|]$*, set* $h_j^{(w)} \leftarrow \mathsf{Hash}(\mathsf{pp}, (h_{2j-1}^{(w+1)}, h_{2j}^{(w+1)}, \mathsf{Val}[v_j^{(w)}]))$*, where* $\mathsf{Val}[v_j^{(w)}]$ *is the index encoded in the* j*-th node of level* w*. If needed, padding is added so that* $|\mathsf{Val}[v_j^{(w)}]| = \lambda$*.*
  - *Let* $\mathsf{m}_1 := \mathsf{h}_{\mathsf{root}}$*, where* $\mathsf{h}_{\mathsf{root}} := h_1^{(0)}$ *is the root hash value. For all* $i \in [|T|]$*,* $w \in \{0, \ldots, d\}$*, and* $j \in [|V^{(w)}|]$*, set* $\mathsf{st} := (\{x_i\}, \{x'_i\}, \{v_j^{(w)}\})$*. Send* m_1 *to* S*.*

$\mathsf{S}(\mathsf{crs}, \mathsf{id}, \mathsf{m}_1)$**:**

- *Parse* $\mathsf{m}_1 := \mathsf{h}_{\mathsf{root}}$ *and* $\mathsf{crs} := \mathsf{pp}$*. Sample* $r, \mathsf{id}'^{(\mathsf{d_m})}, r'^{(\mathsf{d_m})} \xleftarrow{\$} \{0,1\}^\lambda$*. Let* $\mathsf{C}_{\mathsf{d_m}} := \mathsf{F}[\mathsf{id}, \mathsf{id}'^{(\mathsf{d_m})}, r, r'^{(\mathsf{d_m})}]$ *(Fig. 4.1). Garble* $(\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \{\mathsf{lb}_{i,b}^{(\mathsf{d_m})}\}) \xleftarrow{\$} \mathsf{Garb}(\mathsf{C}_{\mathsf{d_m}})$*. For* w *from* $\mathsf{d_m} - 1$ *to* 0 *do:*

  1. *Sample randomness* $\boldsymbol{r}^{(w)}$*, strings* $\mathsf{id}'^{(w)}, r'^{(w)} \xleftarrow{\$} \{0,1\}^\lambda$*, and padding* $\mathsf{pad}^{(w)} \xleftarrow{\$} \{0,1\}^{2(n-1)}$*. Let* $\mathsf{C}_w := \mathsf{V}[\mathsf{pp}, \mathsf{id}, \{\mathsf{lb}_{i,b}^{(w+1)}\}, \boldsymbol{r}^{(w)}, r, \mathsf{id}'^{(w)}, r'^{(w)}, \mathsf{pad}^{(w)}]$*.*
  2. *Garble* $(\widetilde{\mathsf{C}}_w, \{\mathsf{lb}_{i,b}^{(w)}\}) \xleftarrow{\$} \mathsf{Garb}(\mathsf{C}_w)$*.*

Circuit $\mathsf{F}[\mathsf{id}, \mathsf{id}', r, r'](v, x, x')$**:**

HARDWIRED: Target identity id, signal value r, and random strings id', r'.

OPERATION:

1. Sample $\mathsf{pad}, \mathsf{pad}' \xleftarrow{\$} \{0,1\}^{2(n-1)}$
2. Return

$$\begin{cases} \{\mathsf{id}, r, \mathsf{pad}\} & \text{if } v = 1^\lambda \\ \{\mathsf{id}', r', \mathsf{pad}'\} & \text{otherwise} \end{cases}.$$

Circuit $\mathsf{V}[\mathsf{pp}, \mathsf{id}, \{\mathsf{lb}_{i,b}\}, \boldsymbol{r}, r, \mathsf{id}', r', \mathsf{pad}](\alpha, \beta, \gamma)$**:**

HARDWIRED: Hash public parameter pp, target identity id, labels $\{\mathsf{lb}_{i,b}\}$, HEnc randomness $\boldsymbol{r}$, signal value r, random strings id', r', and padding pad.

OPERATION:

1. If $\alpha = 1^\lambda$ then return $\{\mathsf{id}, r, \mathsf{pad}\}$.
2. If $\alpha = 0^\lambda$ then return $\{\mathsf{id}', r', \mathsf{pad}\}$.
3. Else set $h_1 \leftarrow \alpha$, $h_2 \leftarrow \beta$, $i \leftarrow \gamma$ and return

$$\{\mathsf{cth}_{i,b}\} \leftarrow \begin{cases} \mathsf{HEnc}(\mathsf{pp}, h_1, \{\mathsf{lb}_{i,b}\}; \boldsymbol{r}) & \text{if } \mathsf{id}[i] = 0 \\ \mathsf{HEnc}(\mathsf{pp}, h_2, \{\mathsf{lb}_{i,b}\}; \boldsymbol{r}) & \text{otherwise} \end{cases}.$$

Figure 4.1: **Circuits F and V for construction 1. Circuits based on those in Table 1 of [ABD+21].**

- *Let* $\{\mathsf{cth}_{i,b}^{(0)}\} \xleftarrow{\$} \mathsf{HEnc}(\mathsf{pp}, \mathsf{h}_{\mathsf{root}}, \{\mathsf{lb}_{i,b}^{(0)}\})$.
- *Send* $\mathsf{m}_2 := (\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_0, \{\mathsf{cth}_{i,b}^{(0)}\}, r)$ *to* R.

$\mathsf{R}_2(\mathsf{crs}, \mathsf{st}, \mathsf{m}_2)$*:*

- *Parse* $\mathsf{st} := (\{x_i\}, \{x_i'\}, \{v_j^{(w)}\})$ *and* $\mathsf{m}_2 := (\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_0, \{\mathsf{cth}_{i,b}^{(0)}\}, r)$. *For all leaves* $u \in T$, *let* $\mathsf{pth}_u := ((\mathsf{Val}[u]^{\times\lambda}, x, x'), \ldots, \mathsf{h}_{\mathsf{root}})$ *be the root to leaf* u *path in the BP. Let* ℓ *be the length of* pth_u *and let*

$$(\mathsf{id}_u, r_u) \leftarrow \mathsf{DecPath}(\mathsf{pth}_u, \widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_0, \{\mathsf{cth}_{i,b}^{(0)}\}).$$

  If $r_u = r$, *then output* id_u *and halt. If for all* $u \in T$, $r_u \neq r$, *then output* $\perp$.

R_2 must run $\mathsf{DecPath}$ on every root-to-leaf path. R_2 is written above as if there is a unique path from the root to each leaf. But since we allow nodes to have in-degree greater than one, it is possible for a leaf to be reachable from more than one path. In such a case, the path iteration in R_2 should be modified so that all paths are explored.

Communication costs. The first message, m_1, is output by R_1 and sent to S. m_1 contains the hash digest $\mathsf{h}_{\mathsf{root}}$, which is λ bits. So the receiver's communication cost is λ. Next, m_2 is output by S and sent to R_2. m_2 consists of the following:

Procedure $\mathsf{DecPath}(\mathsf{pth}, \mathsf{m}_2)$**:**

INPUT: A leaf-root path pth of length $\ell \leq d$ and tuple $\mathsf{m}_2 := (\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_0, \{\mathsf{cth}_{i,b}^{(0)}\}, r)$.

OPERATION: 1. Parse

$$\mathsf{pth} := ((\underbrace{\mathsf{Val}[v^{(\ell)}]^{\times\lambda}, x, x'}_{z_\ell}), (\underbrace{h^{(\ell)}, h'^{(\ell)}, \mathsf{Val}[v^{(\ell-1)}]}_{z_{\ell-1}}), \ldots, (\underbrace{h^{(1)}, h'^{(1)}, \mathsf{Val}[v^{(0)}]}_{z_0}), \mathsf{h}_{\mathsf{root}}) .$$

2. For w from 0 to $\ell - 1$ do:

   (a) Let $\{\mathsf{lb}_i^{(w)}\} \leftarrow \mathsf{HDec}(z_w, \{\mathsf{cth}_{i,b}^{(w)}\})$. (b) Set $\{\mathsf{cth}_{i,b}^{(w+1)}\} \leftarrow \mathsf{Eval}(\widetilde{\mathsf{C}}_w, \{\mathsf{lb}_i^{(w)}\})$.

3. Let $\{\mathsf{lb}_i^{(\ell)}\} \leftarrow \mathsf{HDec}(z_\ell, \{\mathsf{cth}_{i,b}^{(\ell)}\})$.

4. Let $\{\mathsf{id}_{\mathsf{pth}}, r_{\mathsf{pth}}, \mathsf{pad}\} \leftarrow \mathsf{Eval}(\widetilde{\mathsf{C}}_\ell, \{\mathsf{lb}_i^{(\ell)}\})$ and return $(\mathsf{id}_{\mathsf{pth}}, r_{\mathsf{pth}})$.

Figure 4.2: **Procedure** DecPath **for construction 1. See Fig. A.3 for an illustration of** DecPath**. Based on those in Table 1 of [ABD+21].**

- $\widetilde{\mathsf{C}}_0$: the garbling of circuit F. F has 4λ bits hardcoded (including id).
- $\widetilde{\mathsf{C}}_i$ for $i \in [\mathsf{d_m}]$: each $\widetilde{\mathsf{C}}_i$ is a garbling of circuit V. Hardcoded in each V is the public parameter, $2n = 6\lambda$ garbled circuit labels, hash encryption randomness, and an additional $\mathsf{poly}(\lambda)$ bits (including id).
- $\{\mathsf{cth}_{i,b}^{(0)}\}_{i\in[n],b\in\{0,1\}}$: 6λ hash encryption ciphertexts. Each cth is the hash encryption of one garbled circuit label.
- r: the λ-bit signal string.

So the sender's communication cost grows with $\mathsf{poly}(\lambda, \mathsf{d_m}, |\mathsf{id}|)$, which is $\mathsf{poly}(\lambda)$.

Computation costs.

R_1: performs $|V|$ Hash evaluations and samples $2|T|$ random strings of length λ.

S: samples $\mathsf{poly}(\lambda, \mathsf{d_m})$ random bits, garbles an F circuit, garbles $\mathsf{d_m}$ V circuits, and performs a hash encryption of 6λ garbled labels. The sender's computation cost does not depend on the total size of the BP, just $\mathsf{d_m}$.

R_2: runs $\mathsf{DecPath}$ for every root-leaf path. Each iteration of $\mathsf{DecPath}$ requires at most $\mathsf{d_m}+1$ HDec and garbled circuit Eval evaluations (all, but possibly one, Eval will be of a V circuit).

In total, the receiver's computation cost is $O(\lambda, \mathsf{d_m}, |V|, |\mathsf{PTH}|)$, where $|\mathsf{PTH}|$ is the total number of root-to-leaf paths in the BP. So we require $|V|$ and $|\mathsf{PTH}|$ to be $\mathsf{poly}(\lambda)$ and $\mathsf{d_m} = \lambda + 1$. The sender's computation cost is $\mathsf{poly}(\lambda, \mathsf{d_m})$.

Lemma 3. *Construction 1 is correct in the sense that (1) if* $\mathsf{BP}(\mathsf{id}) = 1$*, then with overwhelming probability* R_2 *outputs* id *and (2) if* $\mathsf{BP}(\mathsf{id}) = 0$*, then with overwhelming probability* R_2 *outputs* $\perp$*.*

Theorem 1. *If* HE *is an anonymous and robust hash encryption (defined in Lemma 1), and* GS *is an anonymous garbling scheme, then the* $\mathsf{BP\text{-}2PC}$ *protocol of Construction 1 provides statistical security for the receiver and semi-honest security for the sender.*

The proofs of Lemma 3 and Theorem 1 are in Sections 4.2 and 4.3, respectively.

4.2 Proof of Lemma 3: correctness of BP-2PC

Proof. (*Proof of Condition (1):* $\mathsf{BP}(\mathsf{id}) = 1 \Rightarrow \mathsf{R}_2$ outputs id with overwhelming probability.)
Claim 1: When $\mathsf{DecPath}$ is evaluated on the correct path, it will output (id, r).
Proof of claim 1: Consider the root-to-leaf path of length ℓ induced by the evaluation of $\mathsf{BP}(\mathsf{id})$. By hypothesis $\mathsf{BP}(\mathsf{id}) = 1$, so the path leaf node encodes the value 1. For concreteness, suppose the induced path has the leftmost leaf of the BP, $u_1 \in T$, as the leaf endpoint. With this in mind, denote the path as,

$$\mathsf{pth}[u_1] := (\underbrace{(1^\lambda, x_1, x_1')}_{z_\ell}, \underbrace{(h_1^{(\ell)}, h_2^{(\ell)}, \mathsf{Val}[v_1^{(\ell-1)}])}_{z_{\ell-1}}, \ldots, \underbrace{(h_1^{(1)}, h_2^{(1)}, \mathsf{Val}[v_1^{(0)}])}_{z_0}, \mathsf{h_{root}})\,. \tag{4.1}$$

For the remainder of the proof, node labels v will be identified with their encoded values $\mathsf{Val}[v]$ to save space. Let $(\mathsf{id}_{u_1}, r_{u_1}) \leftarrow \mathsf{DecPath}(\mathsf{pth}[u_1], \widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_0, \{\mathsf{cth}_{i,b}^{(0)}\})$, where $\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_0, \{\mathsf{cth}_{i,b}^{(0)}\}$ are defined as in the construction. Then it suffices to show that $r_{u_1} = r$. Consider an arbitrary iteration $w \in \{0, \ldots, \ell - 2\}$ of the loop in step 2 of $\mathsf{DecPath}$:

2. (a) $\{\mathsf{lb}_i^{(w)}\} \leftarrow \mathsf{HDec}(z_w, \{\mathsf{cth}_{i,b}^{(w)}\})$
$\leftarrow \mathsf{HDec}((h_1^{(w+1)}, h_2^{(w+1)}, v_1^{(w)}), \mathsf{HEnc}(\mathsf{pp}, h_1^{(w)}, \{\mathsf{lb}_{i,b}^{(w)}\}; \boldsymbol{r}_w))$

$$\leftarrow \mathsf{HDec}(\underbrace{(h_1^{(w+1)}, h_2^{(w+1)}, v_1^{(w)})}, \mathsf{HEnc}(\mathsf{pp}, \mathsf{Hash}(\mathsf{pp}, \underbrace{(h_1^{(w+1)}, h_2^{(w+1)}, v_1^{(w)})}), \{\mathsf{lb}_{i,b}^{(w)}\}; \boldsymbol{r}_w))$$

Since the two terms indicated above are equal, the labels $\{\mathsf{lb}_i^{(w)}\}$ output by HDec are the subset of the encrypted labels $\{\mathsf{lb}_{i,b}^{(w)}\}$ corresponding to the bits of $z_w := (h_1^{(w+1)}, h_2^{(w+1)}, v_1^{(w)})$. More precisely, $\mathsf{lb}_{i,0}^{(w)} := \mathsf{lb}_{i,z_w[i]}^{(w)}$ and $\mathsf{lb}_{i,1}^{(w)} := \mathsf{lb}_{i,z_w[i]}^{(w)}$ for all $i \in [n]$.

2. (b) $\{\mathsf{cth}_{i,b}^{(w+1)}\} \leftarrow \mathsf{Eval}(\widetilde{\mathsf{C}}_w, \{\mathsf{lb}_i^{(w)}\})$.

$$\begin{aligned}
&\{\mathsf{cth}_{i,b}^{(w+1)}\} \leftarrow \mathsf{V}[\mathsf{pp}, \mathsf{id}, \{\mathsf{lb}_{i,b}^{(w+1)}\}, \boldsymbol{r}_w, r, \mathsf{id}', r', \mathsf{pad}](h_1^{(w+1)}, h_2^{(w+1)}, v_1^{(w)}) \\
&\{\mathsf{cth}_{i,b}^{(w+1)}\} \leftarrow \mathsf{HEnc}(\mathsf{pp}, \underbrace{h_1^{(w+1)}}_{=\mathsf{Hash}(\mathsf{pp}, (h_1^{(w+2)}, h_2^{(w+2)}, v_1^{(w+1)}))}, \{\mathsf{lb}_{i,b}^{(w+1)}\}; \boldsymbol{r}_w) \qquad (4.2)
\end{aligned}$$

The first input $h_1^{(w+1)}$ is used in the input to HEnc because $\mathsf{pth}[u_1]$ was defined to have the leftmost leaf as an endpoint. In other words, travelling from the root, $\mathsf{pth}[u_1]$ always progresses to the left child.

In the final iteration of the loop, when $w = \ell - 1$, the steps expanded above remain the same except for Equation 4.2. When $w = \ell - 1$, Eq. 4.2 is instead

$$\{\mathsf{cth}_{i,b}^{(\ell)}\} \leftarrow \mathsf{HEnc}(\mathsf{pp}, \underbrace{h_1^{(\ell)}}_{=\mathsf{Hash}(\mathsf{pp}, (1^\lambda, x_1, x_1'))}, \{\mathsf{lb}_{i,b}^{(\ell)}\}; \boldsymbol{r}_{\ell-1}) .$$

With this in mind, the final two steps of $\mathsf{DecPath}$ are as follows.

3. $\{\mathsf{lb}_i^{(\ell)}\} \leftarrow \mathsf{HDec}(z_\ell, \{\mathsf{cth}_{i,b}^{(\ell)}\})$

$$\{\mathsf{lb}_i^{(\ell)}\} \leftarrow \mathsf{HDec}(\underbrace{(1^\lambda, x_1, x_1')}, \mathsf{HEnc}(\mathsf{pp}, \mathsf{Hash}(\mathsf{pp}, \underbrace{(1^\lambda, x_1, x_1')}), \{\mathsf{lb}_{i,b}^{(\ell)}\})) .$$

Since the two terms indicated above are equal, the labels $\{\mathsf{lb}_i^{(\ell)}\}$ output by HDec are the subset of labels $\{\mathsf{lb}_{i,b}^{(\ell)}\}$ used in the input to HEnc, where the subset corresponds to the bits of $z_\ell = (1^\lambda, x_1, x_1')$.

4. $\{\mathsf{id}_{u_1}, r_{u_1}, \mathsf{pad}\} \leftarrow \mathsf{Eval}\left(\widetilde{\mathsf{C}}_\ell, \{\mathsf{lb}_i^{(\ell)}\}\right)$

$$\begin{aligned}
&\{\mathsf{id}_{u_1}, r_{u_1}, \mathsf{pad}\} \leftarrow \mathsf{V}[\mathsf{pp}, \mathsf{id}, \{\mathsf{lb}_{i,b}^{(\ell+1)}\}, \boldsymbol{r}_\ell, r, \mathsf{id}', r', \mathsf{pad}]\left(1^\lambda, x_1, x_1'\right) \\
&\{\mathsf{id}_{u_1}, r_{u_1}, \mathsf{pad}\} \leftarrow \{\mathsf{id}_{u_1} \leftarrow \mathsf{id}, r_{u_1} \leftarrow r, \text{ and } \mathsf{pad} \xleftarrow{\$} \{0,1\}^{2(n-1)}\} .
\end{aligned}$$

Then $(\mathsf{id}_{u_1}, r_{u_1})$ is returned to the receiver. The first input to V is 1^λ, so the tuple $(\mathsf{id}_{u_1}, r_{u_1})$ is equal to (id, r).

The receiver compares r_{u_1} from $\mathsf{DecPath}$ with r from the message m_2 from the sender. Since these strings are equal, the receiver takes id_{u_1} output from $\mathsf{DecPath}$ as the sender's element. Hence, the receiver learns id when $\mathsf{BP}(\mathsf{id}) = 1$, completing the proof of claim 1. $\diamond$

In the above, we used the correctness properties of garbled circuit evaluation and HE decryption. These guarantees give us that $\Pr[\mathsf{id}_{u_1} = \mathsf{id} \wedge r_{u_1} = r \mid (\mathsf{id}_{u_1}, r_{u_1}) \leftarrow \mathsf{DecPath}(\mathsf{pth}[u_1], \mathsf{m}_2)] = 1$ when $\mathsf{pth}[u_1]$ is the correct path through the BP. In order for the correctness condition (1) to be met, it must also be true that there does not exist any other path $\mathsf{pth}[u'] \neq \mathsf{pth}[u_1]$ such that $r_{u'} = r$ where $(\mathsf{id}_{u'}, r_{u'}) \leftarrow \mathsf{DecPath}(\mathsf{pth}[u'], \mathsf{m}_2)$. In other words, there must not exist an incorrect path that decrypts the correct signal value r.

Claim 2: With at most negligible probability, there exists an incorrect path that when input to $\mathsf{DecPath}$, decrypts to the correct signal value r.
Proof of claim 2: To show that occurs with negligible probability, consider running $\mathsf{DecPath}$ on an incorrect path $\mathsf{pth}[u'] \neq \mathsf{pth}[u_1]$. Let $\mathsf{pth}[u_1]$ and $\mathsf{pth}[u']$ have lengths ℓ and ℓ', respectively where $1 \leq \ell, \ell' \leq d$. Suppose these paths are equal at level $\alpha - 1$ and differ at level α, for some $\alpha \in \{0, \ldots, \min\{\ell, \ell'\}\}$. Suppose $u_1 \in T$ is the leftmost leaf, as above, and $u' \in T \setminus \{u_1\}$ is the leaf endpoint of $\mathsf{pth}[u']$. Let these paths be given by the following.

$$\begin{aligned}\mathsf{pth}[u_1] := (&(\underbrace{u_1^{(\ell)\times\lambda}, x_1, x_1'}_{z_\ell}), (\underbrace{h_1^{(\ell)}, h_2^{(\ell)}, v_1^{(\ell-1)}}_{z_{\ell-1}}), \ldots, (\underbrace{h_1^{(\alpha+1)}, h_2^{(\alpha+1)}, v_1^{(\alpha)}}_{z_\alpha}), \\ &(\underbrace{h_1^{(\alpha)}, h_2^{(\alpha)}, v_1^{(\alpha-1)}}_{z_{\alpha-1}}), \ldots, (\underbrace{h_1^{(1)}, h_2^{(1)}, v_1^{(0)}}_{z_0}), \mathsf{h}_{\mathsf{root}})\end{aligned} \tag{4.3}$$

$$\begin{aligned}\mathsf{pth}[u'] := (&(\underbrace{u'^{(\ell')\times\lambda}, x, x'}_{z'_{\ell'}}), (\underbrace{h^{(\ell')}, h'^{(\ell')}, v^{(\ell'-1)}}_{z'_{\ell'-1}}), \ldots, (\underbrace{h_3^{(\alpha+1)}, h_4^{(\alpha+1)}, v_2^{(\alpha)}}_{z'_\alpha}), \\ &(\underbrace{h_1^{(\alpha)}, h_2^{(\alpha)}, v_1^{(\alpha-1)}}_{z'_{\alpha-1}}), \ldots, (\underbrace{h_1^{(1)}, h_2^{(1)}, v_1^{(0)}}_{z'_0}), \mathsf{h}_{\mathsf{root}})\,.\end{aligned} \tag{4.4}$$

Since $\mathsf{pth}[u']$ differs from the correct path at level α, the steps of $\mathsf{DecPath}(\mathsf{pth}[u'], \mathsf{m}_2)$ and $\mathsf{DecPath}(\mathsf{pth}[u_1], \mathsf{m}_2)$ will be identical until loop iteration $w = \alpha$. Consider iteration $w = \alpha - 1$ of $\mathsf{DecPath}(\mathsf{pth}[u'], \mathsf{m}_2)$:

2. (a) $\{\mathsf{lb}_i^{(\alpha-1)}\} \leftarrow \mathsf{HDec}(z'_{\alpha-1}, \{\mathsf{cth}_{i,b}^{(\alpha-1)}\})$

$$\begin{aligned}
&\leftarrow \mathsf{HDec}((h_1^{(\alpha)}, h_2^{(\alpha)}, v_1^{(\alpha-1)}), \mathsf{HEnc}(\mathsf{pp}, h_1^{(\alpha-1)}, \{\mathsf{lb}_{i,b}^{(\alpha-1)}\})) \\
&\leftarrow \mathsf{HDec}((\underbrace{h_1^{(\alpha)}, h_2^{(\alpha)}, v_1^{(\alpha-1)}}), \mathsf{HEnc}(\mathsf{pp}, \mathsf{Hash}(\mathsf{pp}, (\underbrace{h_1^{(\alpha)}, h_2^{(\alpha)}, v_1^{(\alpha-1)}})), \{\mathsf{lb}_{i,b}^{(\alpha-1)}\}))\,.
\end{aligned}$$

Since the indicated terms are equal, the $\{\mathsf{lb}_i^{(\alpha-1)}\}$ labels output are the labels of circuit $\widetilde{\mathsf{C}}_{\alpha-1}$ corresponding to the bits of $z'_{\alpha-1}$.

2. (b) $\{\mathsf{cth}_{i,b}^{(\alpha)}\} \leftarrow \mathsf{Eval}(\widetilde{\mathsf{C}}_{\alpha-1}, \{\mathsf{lb}_i^{(\alpha-1)}\})$

$$\begin{aligned}
&\leftarrow \mathsf{V}[\mathsf{pp}, \mathsf{id}, \{\mathsf{lb}_{i,b}^{(\alpha)}\}, \boldsymbol{r}_{\alpha-1}, r, \mathsf{id}', r', \mathsf{pad}](h_1^{(\alpha)}, h_2^{(\alpha)}, v_1^{(\alpha-1)}) \\
&\leftarrow \mathsf{HEnc}(\mathsf{pp}, h_1^{(\alpha)}, \{\mathsf{lb}_{i,b}^{(\alpha)}\}; \boldsymbol{r}_{\alpha-1})\,.
\end{aligned}$$

In the last line, $h_1^{(\alpha)}$ is used in the hash encryption due to the assumption that the correct path has the leftmost leaf as an endpoint, meaning $\mathsf{id}[v_1^{(\alpha+1)}] = 0$.[1] Next, the $w = \alpha$ iteration of the loop proceeds as follows.

2. (a) $\{\mathsf{lb}_i'^{(\alpha)}\} \leftarrow \mathsf{HDec}(z'_{\alpha}, \{\mathsf{cth}_{i,b}^{(\alpha)}\})$

$$\begin{aligned}
&\leftarrow \mathsf{HDec}((h_3^{(\alpha+1)}, h_4^{(\alpha+1)}, v_2^{(\alpha)}), \mathsf{HEnc}(\mathsf{pp}, h_1^{(\alpha)}, \{\mathsf{lb}_{i,b}^{(\alpha)}\}; \boldsymbol{r}_{\alpha-1})) \\
&\leftarrow \mathsf{HDec}((\underbrace{h_3^{(\alpha+1)}, h_4^{(\alpha+1)}, v_2^{(\alpha)}}), \mathsf{HEnc}(\mathsf{pp}, (\underbrace{h_1^{(\alpha+1)}, h_2^{(\alpha+1)}, v_1^{(\alpha)}}), \{\mathsf{lb}_{i,b}^{(\alpha)}\}; \boldsymbol{r}_{\alpha-1}))\,.
\end{aligned}$$

The two terms indicated are not equal. Decrypting an HE ciphertext with an incorrect hash pre-image produces an output containing no PPT-accessible information about the encrypted plaintext. For this reason, a prime was added above to the LHS labels to differentiate them from the labels encrypted on the RHS. Thus $\{\mathsf{lb}_i'^{(\alpha)}\}$ provides no information about $\{\mathsf{lb}_{i,b}^{(\alpha)}\}$.

2. (b) $\{\mathsf{cth}_{i,b}^{(\alpha+1)}\} \leftarrow \mathsf{Eval}(\widetilde{\mathsf{C}}_{\alpha}, \{\mathsf{lb}_i'^{(\alpha)}\})$.

Note that the labels $\{\mathsf{lb}_i'^{(\alpha)}\}$ are *not* labels of $\widetilde{\mathsf{C}}_{\alpha}$, and certainly not a subset of those labels corresponding to a meaningful input. So the output $\{\mathsf{cth}_{i,b}^{(\alpha+1)}\}$ is not a ciphertext, but a meaningless set of strings.

[1] Changing the proof to apply to settings with a different correct path is straightforward.

For w from α to ℓ', every HDec operation will output $\{\mathsf{lb}_i'^{(w)}\}$ which are *not* circuit labels for $\widetilde{\mathsf{C}}_w$ and every evaluation $\mathsf{Eval}(\widetilde{\mathsf{C}}_w, \{\mathsf{lb}_i'^{(w)}\})$ will output strings with no relation to $\widetilde{\mathsf{C}}_w$. In step 4, $\{\mathsf{id}_{u'}, r_{u'}, \mathsf{pad}\} \leftarrow \mathsf{Eval}(\widetilde{\mathsf{C}}_{\ell'}, \{\mathsf{lb}_i'^{(\ell')}\})$ is computed. Since $\{\mathsf{lb}_i'^{(\ell')}\}$ are not labels, the evaluation output is meaningless. In particular, the tuple $(\mathsf{id}_{u'}, r_{u'})$ output to R_2 contains no PPT-accessible information about (id, r). Hence $\Pr[r_{u'} = r] \leq 2^{-\lambda} + \mathsf{negl}(\lambda)$. By assumption on the size of BP, there are a polynomial number of root-to-leaf paths, thus by the union bound the probability that any incorrect root-to-path causes $\mathsf{DecPath}$ to output r is

$$\Pr[\exists\ u \in T \setminus \{u_1\} \text{ s.t. } r_u = r \mid (\mathsf{id}_u, r_u) \leftarrow \mathsf{DecPath}(\mathsf{pth}[u], \mathsf{m}_2)] \leq \frac{\mathsf{poly}(\lambda)}{2^\lambda} + \mathsf{negl}(\lambda)\,.$$

The probability that R_2 outputs id when $\mathsf{BP}(\mathsf{id}) = 1$ is the probability that none of the incorrect paths output a signal value equal to r:

$$\Pr[\mathsf{R}_2 \text{ outputs } \mathsf{id} \mid \mathsf{BP}(\mathsf{id}) = 1] \geq 1 - \frac{\mathsf{poly}(\lambda)}{2^\lambda} - \mathsf{negl}(\lambda)\,.$$

Thus proving Claim 2. ◇

Taken with the proof of Claim 1, this completes the proof of correctness condition (1). □

Proof. (*Proof of Condition (2):* $\mathsf{BP}(\mathsf{id}) = 0 \Rightarrow \mathsf{R}_2$ outputs $\perp$ with overwhelming probability.)
In the proof of Theorem 1 we will show that when $\mathsf{BP}(\mathsf{id}) = 0$,

$$(\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_0, \{\mathsf{cth}_{i,b}^{(0)}\}, r) \stackrel{c}{\equiv} (\widetilde{\mathsf{C}}'_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}'_0, \{\mathsf{cth}_{i,b}'^{(0)}\}, r')\,, \tag{4.5}$$

where all primed values are sampled uniformly random. On the LHS, the circuits $\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_0$ all have the signal value r hardcoded, while the RHS is independent of r. So, for all fixed $u \in T$,

$$\Pr\Big[r_u = r \mid (\mathsf{id}_u, r_u) \leftarrow \mathsf{DecPath}(\mathsf{pth}[u], (\widetilde{\mathsf{C}}'_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}'_0, \{\mathsf{cth}_{i,b}'^{(0)}\}, r'))\Big] \leq \frac{1}{2^\lambda}\,, \tag{4.6}$$

where $\mathsf{pth}[u]$ denotes the path from the root to leaf u[2]. By assumption, the BP has a polynomial number of root-to-leaf paths, thus by the union bound the probability that any

[2]There could exist a $\mathsf{poly}(\lambda)$ number of root-to-leaf u paths. In such a case, the RHS of Eq. 4.6 would be $\mathsf{poly}(\lambda)/2^\lambda$ and Eq. 4.7 would remain unchanged.

root-to-leaf paths decrypt to output r is,

$$\Pr\Big[\exists\, u \in T \text{ s.t. } r_u = r \mid (\mathsf{id}_u, r_u) \leftarrow \mathsf{DecPath}(\mathsf{pth}[u], (\widetilde{\mathsf{C}}'_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}'_0, \{\mathsf{cth}'^{(0)}_{i,b}\}, r'))\Big] \leq \frac{\mathsf{poly}(\lambda)}{2^\lambda}\,. \tag{4.7}$$

By Equation 4.5, we must also have that the analogous probability for inputs $(\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_0, \{\mathsf{cth}^{(0)}_{i,b}\}, r)$ is computationally indistinguishable. Thus,

$$\Pr\Big[\exists\, u \in T \text{ s.t. } r_u = r \mid (\mathsf{id}_u, r_u) \leftarrow \mathsf{DecPath}(\mathsf{pth}[u], (\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_0, \{\mathsf{cth}^{(0)}_{i,b}\}, r))\Big] \leq \frac{\mathsf{poly}(\lambda)}{2^\lambda} + \mathsf{negl}(\lambda)\,.$$

If R_2 receives an r_u from $\mathsf{DecPath}$ such that $r_u = r$, then R_2 outputs id_u, not $\perp$. It directly follows that,

$$\Pr[\mathsf{R}_2 \text{ does not output } \perp \mid \mathsf{BP}(\mathsf{id}) = 0] \leq \frac{\mathsf{poly}(\lambda)}{2^\lambda} + \mathsf{negl}(\lambda)$$
$$\Pr[\mathsf{R}_2 \text{ outputs } \perp \mid \mathsf{BP}(\mathsf{id}) = 0] \geq 1 - \frac{\mathsf{poly}(\lambda)}{2^\lambda} - \mathsf{negl}(\lambda)\,,$$

which completes the proof of Condition (2), and thus concludes the proof of Lemma 3. □

4.3 Proof of Theorem 1: security of BP-2PC

Theorem 1 statistical receiver security proof. Node labels are identified with their encoded values to save space. Following Definition 7, for any pair $(\mathrm{BP}_0, \mathrm{BP}_1)$ consider the distribution below for $i \in \{0, 1\}$.

$$\begin{aligned}\mathsf{view}_S^{\mathsf{BP\text{-}2PC}}(\mathrm{BP}_i, \mathsf{id}, \lambda) &= (\mathsf{id}, \boldsymbol{r}_\mathsf{S}, \mathsf{m}_1, \mathsf{m}_2)\\ &= (\mathsf{id}, \boldsymbol{r}_\mathsf{S}, (\mathsf{d_m}, \mathsf{h}_{\mathsf{root}i}), (\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_0, \mathsf{HEnc}(\mathsf{pp}, \mathsf{h}_{\mathsf{root}i}, \{\mathsf{lb}^{(0)}_{i,b}\}), r))\,,\end{aligned}$$

where $\boldsymbol{r}_\mathsf{S}$ are the sender's random coins, $\mathsf{h}_{\mathsf{root}i}$ is the root hash, and $\mathsf{d_m}$ is the maximum depth of branching program BP_i. Since both BPs have security parameter λ, both will have $\mathsf{d_m} = \lambda + 1$. Let d_i be the depth of BP_i.

Robustness of HE implies that for all $\mathsf{pp} \xleftarrow{\$} \mathsf{HGen}(1^\lambda, 3\lambda)$ and $u \in T$, the distribu-

tion $(\mathsf{pp}, \mathsf{Hash}(\mathsf{pp}, (u^\lambda, x, x')))$, where $x, x' \xleftarrow{\$} \{0,1\}^\lambda$, is statistically close to $(\mathsf{pp}, h_\$)$ where $h_\$ \xleftarrow{\$} \{0,1\}^\lambda$. Hence $\mathsf{Hash}(\mathsf{pp}, (u^\lambda, x, x'))$ statistically hides u. At level d_i, BP_i will have at least two leaf nodes with the same parent. Let u_1, u_2 be two such leaves and let $v^{(d_i-1)}$ be the parent. Node $v^{(d_i-1)}$ will then have hash value,

$$\begin{aligned} h^{(d_i-1)} &\leftarrow \mathsf{Hash}(\mathsf{pp}, (h_1^{(d_i)}, h_2^{(d_i)}, v^{(d_i-1)})) \\ &\leftarrow \mathsf{Hash}(\mathsf{pp}, (\mathsf{Hash}(\mathsf{pp}, (u_1^\lambda, x_1, x_1')), \mathsf{Hash}(\mathsf{pp}, (u_2^\lambda, x_2, x_2')), v^{(d_i-1)})) \,. \end{aligned}$$

Since $h_1^{(d_i)}$ and $h_2^{(d_i)}$ are both statistically close to uniform, we have that $h^{(d_i-1)}$ is also statistically close to uniform. Continuing up the tree in this way, we see that the root hash $\mathsf{h}_{\mathsf{root}i}$ is also indistinguishable from random and thus statistically hides BP. Thus $\mathsf{h}_{\mathsf{root}0} \approx_s \mathsf{h}_{\mathsf{root}1}$, which gives us $\mathsf{view}_S^{\mathsf{BP\text{-}2PC}}(\mathrm{BP}_0, \mathsf{id}, \lambda) \approx_s \mathsf{view}_S^{\mathsf{BP\text{-}2PC}}(\mathrm{BP}_1, \mathsf{id}, \lambda)$. □

Theorem 1 semi-honest sender security proof. Sender security will be proved through a sequence of indistinguishable hybrids in two main steps. First, all garbled circuits in the sender's message m_2 are replaced one at a time with simulated circuits. Then m_2 is switched to random.

Sender security only applies when $\mathsf{BP}(\mathsf{id}) = 0$, so this will be assumed for the proof. For concreteness, suppose the path induced on the BP by evaluating id has the leftmost leaf as an endpoint. In particular, let

$$\mathsf{pth} := (\underbrace{(\mathsf{Val}[v_1^{(\ell)}]^{\times\lambda}, x_1, x_1')}_{z_\ell}, \underbrace{(h_1^{(\ell)}, h_2^{(\ell)}, \mathsf{Val}[v_1^{(\ell-1)}])}_{z_{\ell-1}}, \ldots, \underbrace{(h_1^{(1)}, h_2^{(1)}, \mathsf{Val}[v_1^{(0)}])}_{z_0}, \mathsf{h}_{\mathsf{root}}) \tag{4.8}$$

be the leaf-root path induced on the hashed BP by evaluation of id, where ℓ is the path length and d is the BP depth.[3] Since $\mathsf{BP}(\mathsf{id}) = 0$, the terminal node encodes value 0, i.e, $\mathsf{Val}[v_1^{(\ell)}] = 0$. We let $\mathsf{h}_{\mathsf{root}} \leftarrow \mathsf{Hash}(\mathsf{pp}, z_0)$ and $h_1^{(i)} \leftarrow \mathsf{Hash}(\mathsf{pp}, z_i)$ for all $1 \le i \le \ell$, where the z_i values are defined as in Eq. 4.8. To save space, often node labels v will be identified with their encoded index values $\mathsf{Val}[v]$ and the padding superscript will be omitted from leaf node values.

$\mathbf{Hyb}_0$: [Fig. 4.3 left] The sender's message $\mathsf{m}_2 := (\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_0, \{\mathsf{cth}_{i,b}^{(0)}\}, r)$ is formed as described in the construction.

$\mathbf{Hyb}_1$: [Fig. 4.3 right] All circuits are simulated. The circuits are simulated so that if R runs $\mathsf{DecPath}$ on pth with the simulated circuits, then every step occurs, from the view

[3] We assume $\ell \ge 1$. If the receiver's BP has depth 0, then two dummy leaves can be introduced as root children.

of R, as it would in $\mathbf{Hyb}_0$. This requires knowledge of pth and in particular, the correct sequence of hash pre-images $z_\ell, \ldots, z_0$, where $z_\ell = (0^\lambda, x_1, x'_1)$ and $z_j = (h_1^{(j+1)}, h_2^{(j+1)}, v_1^{(j)})$ for $j \in \{0, \ldots, \ell - 1\}$. By assumption of pth, every evaluation $\mathsf{Eval}(\widetilde{\mathsf{C}}_j, \{\mathsf{lb}_i^{(j)}\})$, where $\{\mathsf{lb}_i^{(j)}\} \leftarrow \mathsf{HDec}(z_j, \{\mathsf{cth}_{i,b}^{(j)}\})$, done in $\mathsf{DecPath}$ for $j \in \{0, \ldots, \ell - 1\}$ will output ciphertexts $\mathsf{HEnc}(\mathsf{pp}, h_1^{(j+1)}, \{\mathsf{lb}_{i,b}^{(j+1)}\}; \boldsymbol{r}_j)$[4]. Moreover, evaluation of $\mathsf{Eval}(\widetilde{\mathsf{C}}_\ell, \{\mathsf{lb}_i^{(\ell)}\})$ outputs $\{\mathsf{id}', r', \mathsf{pad}\}$ for random $\mathsf{id}', r' \xleftarrow{\$} \{0,1\}^\lambda$ and $\mathsf{pad} \xleftarrow{\$} \{0,1\}^{2(n-1)}$. Simulating circuits $\widetilde{\mathsf{C}}_\ell, \ldots, \widetilde{\mathsf{C}}_0$ is straightforward.

To simulate circuits $\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_{\ell+1}$ we note that none of these circuits can be used by R in $\mathsf{DecPath}$ to obtain a meaningful output. Only this behaviour needs to be mimicked. To this end, we define "ghost" values $z_{\mathsf{d_m}}, \ldots, z_{\ell+1}$ with their associated hash values. The deepest is defined to be uniformly random: $z_{\mathsf{d_m}} \xleftarrow{\$} \{0,1\}^{3\lambda}$. Then for $j \in \{\mathsf{d_m} - 1, \ldots, \ell + 1\}$ we define,

$$h'^{(j)} := \mathsf{Hash}(\mathsf{pp}, (\overbrace{\underbrace{h'^{(j+1)}}_{\mathsf{Hash}(\mathsf{pp}, z_{j+1})}, h'^{(j+1)}, v'^{(j)}}^{z_j}))$$

where $v'_j \xleftarrow{\$} \{0,1\}^\lambda$. In this way, two-thirds of the z_j pre-image is uniformly random which allows us to invoke the robustness property of HE. Moreover, the z_j values create a chain of pre-images similar to the z_j values for $0 \le j \le \ell - 1$.

Lemma 4. *Hybrids* $\mathbf{Hyb}_0$ *and* $\mathbf{Hyb}_1$ *are computationally indistinguishable.*

$\mathbf{Hyb}_2$: Sample m_2 at random.

Lemma 5. *Hybrids* $\mathbf{Hyb}_1$ *and* $\mathbf{Hyb}_2$ *are computationally indistinguishable.*

If m_2 is pseudorandom to the receiver, then m_2 created with some id_0 is computationally indistinguishable from m_2 created with some other id_1. Therefore we have $\mathsf{view}_R^{\mathsf{BP\text{-}2PC}}(\mathrm{BP}, \mathsf{id}_0, \lambda) \stackrel{c}{\equiv} \mathsf{view}_R^{\mathsf{BP\text{-}2PC}}(\mathrm{BP}, \mathsf{id}_1, \lambda)$, hence the above two lemmas establish computational sender security.

[4]The use of $h_1^{(j+1)}$ in HEnc is from the assumption that pth has the leftmost leaf as an endpoint and hence the first hash input is always used in the V encryption. In the general case, this hash value would be changed accordingly.

$\mathbf{Hyb}_0$:

$r, r', \mathsf{id}' \xleftarrow{\$} \{0,1\}^\lambda$; $\mathsf{pad} \xleftarrow{\$} \{0,1\}^{2(n-1)}$

$\mathsf{C}_{\mathsf{d_m}} := \mathsf{F}[\mathsf{id}, \mathsf{id}', r, r']$

$(\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \{\mathsf{lb}_{i,b}^{(\mathsf{d_m})}\}) \xleftarrow{\$} \mathsf{Garb}(\mathsf{C}_{\mathsf{d_m}})$

For w from $\mathsf{d_m} - 1$ to 0 do

  Sample random $\boldsymbol{r}_w$

  $\mathsf{C}_w := \mathsf{V}[\mathsf{pp}, \mathsf{id}, \{\mathsf{lb}_{i,b}^{(w+1)}\}, \boldsymbol{r}_w, r, \mathsf{id}', r', \mathsf{pad}]$

  $(\widetilde{\mathsf{C}}_w, \{\mathsf{lb}_{i,b}^{(w)}\}) \xleftarrow{\$} \mathsf{Garb}(\mathsf{C}_w)$

$\{\mathsf{cth}_{i,b}^{(0)}\} \xleftarrow{\$} \mathsf{HEnc}(\mathsf{pp}, \mathsf{h}_{\mathsf{root}}, \{\mathsf{lb}_{i,b}^{(0)}\})$

$\mathsf{m}_2 := (\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_0, \{\mathsf{cth}_{i,b}^{(0)}\}, r)$

Return m_2

$\mathbf{Hyb}_1$:

$r, r', \mathsf{id}' \xleftarrow{\$} \{0,1\}^\lambda$; $\mathsf{pad} \xleftarrow{\$} \{0,1\}^{2(n-1)}$

$z_{\mathsf{d_m}} \xleftarrow{\$} \{0,1\}^{3\lambda}$; $(\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \{\mathsf{lb}_i^{(\mathsf{d_m})}\}) \xleftarrow{\$} \mathsf{Sim}(\mathsf{F}, \{\mathsf{id}', r'\})$

For $0 \le w \le \mathsf{d_m} - 1$ sample random $\boldsymbol{r}_w$

For i from $\mathsf{d_m} - 1$ down to $\ell + 1$ do

  $v'^{(i)} \xleftarrow{\$} \{0,1\}^\lambda$; $h'^{(i+1)} \leftarrow \mathsf{Hash}(\mathsf{pp}, z_{i+1})$

  $z_i := (h'^{(i+1)}, h'^{(i+1)}, v'^{(i)})$

For w from $\mathsf{d_m} - 1$ down to $\ell + 1$ do

  $\{\mathsf{cth}_{i,b}^{(w+1)}\} \leftarrow \mathsf{HEnc}(\mathsf{pp}, h'^{(w+1)}, \{\mathsf{lb}_{i,b}^{(w+1)}\}; \boldsymbol{r}_w)$

  $(\widetilde{\mathsf{C}}_w, \{\mathsf{lb}_i^{(w)}\}) \xleftarrow{\$} \mathsf{Sim}(\mathsf{V}, \{\mathsf{cth}_{i,b}^{(w+1)}\})$

$(\widetilde{\mathsf{C}}_\ell, \{\mathsf{lb}_i^{(\ell)}\}) \xleftarrow{\$} \mathsf{Sim}(\mathsf{V}, \{\mathsf{id}', r', \mathsf{pad}\})$

For w from $\ell - 1$ down to 0 do

  $\{\mathsf{cth}_{i,b}^{(w+1)}\} \leftarrow \mathsf{HEnc}(\mathsf{pp}, h_1^{(w+1)}, \{\mathsf{lb}_i^{(w+1)}\}; \boldsymbol{r}_w)$

  $(\widetilde{\mathsf{C}}_w, \{\mathsf{lb}_i^{(w)}\}) \xleftarrow{\$} \mathsf{Sim}(\mathsf{V}, \{\mathsf{cth}_{i,b}^{(w+1)}\})$

$\{\mathsf{cth}_{i,b}^{(0)}\} \xleftarrow{\$} \mathsf{HEnc}(\mathsf{pp}, \mathsf{h}_{\mathsf{root}}, \{\mathsf{lb}_i^{(0)}\})$

Return $\mathsf{m}_2 := (\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_0, \{\mathsf{cth}_{i,b}^{(0)}\}, r)$

Figure 4.3: **$\mathbf{Hyb}_0$ and $\mathbf{Hyb}_1$ for the proof of Theorem 1.**

4.3.1 Proof of Lemma 4

To prove that $\mathbf{Hyb}_0 \overset{c}{\equiv} \mathbf{Hyb}_1$, we define a chain of $\mathsf{d_m} + 1$ hybrids between $\mathbf{Hyb}_0$ and $\mathbf{Hyb}_1$. Then we prove each game hop is indistinguishable.

$\mathbf{Hyb}_{1.p}$ *for* $0 \le p \le \mathsf{d_m}$ *(Fig. 4.5):* Let pth be as in Equation 4.8 and recall we assume that $\mathsf{Val}[v_1^{(\ell)}] = 0$. In $\mathbf{Hyb}_{1.p}$ circuits $\widetilde{\mathsf{C}}_0, \ldots, \widetilde{\mathsf{C}}_{p-1}$ are simulated and circuits $\widetilde{\mathsf{C}}_p, \ldots, \widetilde{\mathsf{C}}_{\mathsf{d_m}}$ are honestly generated (as in $\mathbf{Hyb}_0$). In $\mathbf{Hyb}_{1.0}$, all circuits are generated honestly[5] and in $\mathbf{Hyb}_{1.\mathsf{d_m}}$ all circuits are simulated except for $\widetilde{\mathsf{C}}_{\mathsf{d_m}}$.

The way a particular circuit $\widetilde{\mathsf{C}}_i$ for $i \le p - 1$ is simulated depends on if $i < \ell$, $i = \ell$, or $i > \ell$, where ℓ is the length of path induced by id. These differences are shown in Fig. 4.4. As in $\mathbf{Hyb}_1$, simulating circuits $\widetilde{\mathsf{C}}_{\ell+1}, \ldots, \widetilde{\mathsf{C}}_{\mathsf{d_m}-1}$ is done using ciphertexts created with "ghost" z values.

[5]When $p = 0$, $\mathbf{Hyb}_{1.p}$ is defined so that circuits $\widetilde{\mathsf{C}}_0, \ldots, \widetilde{\mathsf{C}}_{-1}$ are simulated, which we define to mean that no circuits are simulated.

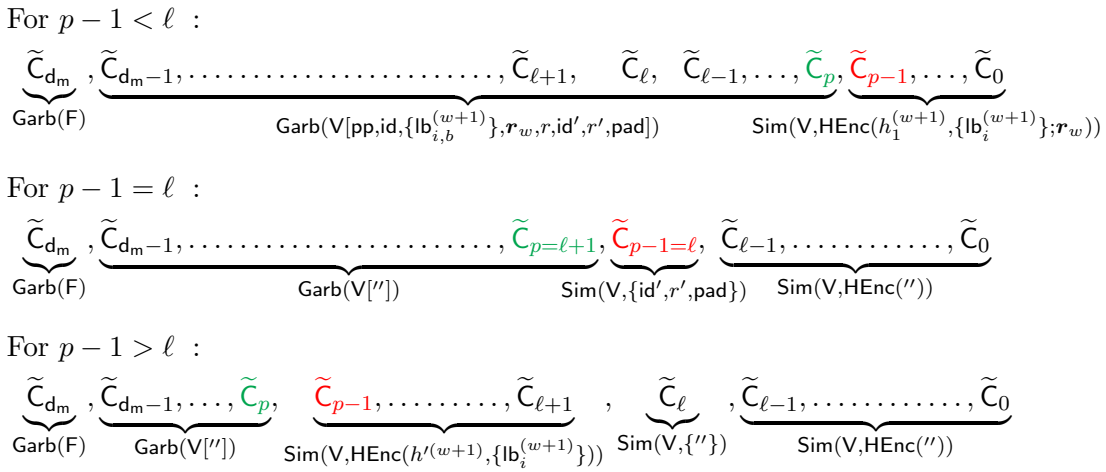


Figure 4.4: **Method of generating circuits in $\mathbf{Hyb}_{1.p}$ depending on the value of $p-1$ relative to the value of ℓ. Use of $h_1^{(w+1)}$ in HEnc on the LHS is from the assumption that pth has the leftmost leaf as an endpoint. $''$ is the ditto symbol.**

Lemma 6. $\mathbf{Hyb}_0 \overset{c}{\equiv} \mathbf{Hyb}_{1.0}$ *and* $\mathbf{Hyb}_1 \overset{c}{\equiv} \mathbf{Hyb}_{1.\mathsf{d_m}}$.

Proof. First we will prove $\mathbf{Hyb}_0 \overset{c}{\equiv} \mathbf{Hyb}_{1.0}$ (Fig. 4.3 and Fig. 4.5). In both hybrids all circuits are honestly generated, but they differ in two ways. The first is in how the labels $\{\mathsf{lb}^{(\mathsf{d_m})}\}$ are formed. Both hybrids generate the tuple $(\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \{\mathsf{lb}_{i,b}^{(\mathsf{d_m})}\}) \xleftarrow{\$} \mathsf{Garb}(\mathsf{F}[\mathsf{id}, \mathsf{id}', r, r'])$ but $\mathbf{Hyb}_{1.0}$ additionally does $\{\mathsf{lb}_i^{(\mathsf{d_m})}\} := \{\mathsf{lb}_{i,z_{\mathsf{d_m}}[i]}^{(\mathsf{d_m})}\}$. If $\ell < \mathsf{d_m}$, then $z_{\mathsf{d_m}}$ is random. In that case, $\mathsf{Eval}(\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \{\mathsf{lb}_{i,z_{\mathsf{d_m}}[i]}^{(\mathsf{d_m})}\})$ will return $\{\mathsf{id}', r'\}$ with overwhelming probability. If $\ell = \mathsf{d_m}$ then $z_{\mathsf{d_m}} := (0^\lambda, x_1, x_1')$ and so $\mathsf{Eval}(\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \{\mathsf{lb}_{i,z_{\mathsf{d_m}}[i]}^{(\mathsf{d_m})}\})$ will return $\{\mathsf{id}', r'\}$ with probability 1. Hence the difference between the sets of labels is indistinguishable by the $\mathsf{BP}(\mathsf{id}) = 0$ assumption.

The second difference between $\mathbf{Hyb}_0$ and $\mathbf{Hyb}_{1.0}$ is in how $\{\mathsf{cth}_{i,b}^{(0)}\}$ is formed. In $\mathbf{Hyb}_0$ we define $\{\mathsf{cth}_{i,b}^{(0)}\} \xleftarrow{\$} \mathsf{HEnc}(\mathsf{pp}, \mathsf{h}_{\mathsf{root}}, \{\mathsf{lb}_{i,b}^{(0)}\})$. While $\mathbf{Hyb}_{1.0}$ does $\{\mathsf{cth}_{i,b}^{(0)}\} \xleftarrow{\$} \mathsf{HEnc}(\mathsf{pp}, \mathsf{h}_{\mathsf{root}}, \{\mathsf{lb}_i^{(0)}\})$, where $\{\mathsf{lb}_i^{(0)}\} := \{\mathsf{lb}_{i,z_0[i]}^{(0)}\}$. Since $\mathsf{h}_{\mathsf{root}} \leftarrow \mathsf{Hash}(\mathsf{pp}, z_0)$, by semantic security of hash encryption we have that $\mathsf{HEnc}(\mathsf{pp}, \mathsf{h}_{\mathsf{root}}, \{\mathsf{lb}_i^{(0)}\}) \overset{c}{\equiv} \mathsf{HEnc}(\mathsf{pp}, \mathsf{h}_{\mathsf{root}}, \{\mathsf{lb}_{i,b}^{(0)}\})$, completing the proof of $\mathbf{Hyb}_0 \overset{c}{\equiv} \mathbf{Hyb}_{1.0}$.

Next we will prove $\mathbf{Hyb}_1 \overset{c}{\equiv} \mathbf{Hyb}_{1.\mathsf{d_m}}$. In $\mathbf{Hyb}_1$ (Fig. 4.3), all circuits are simulated. In $\mathbf{Hyb}_{1.\mathsf{d_m}}$ (Fig. 4.5), all circuits are simulated except for $\widetilde{\mathsf{C}}_{\mathsf{d_m}}$. The two hybrids are the

$\mathbf{Hyb}_{1.p}$:

$r, \mathsf{id}', r' \xleftarrow{\$} \{0,1\}^{\lambda}$; $\mathsf{pad} \xleftarrow{\$} \{0,1\}^{2(n-1)}$; $z_{\mathsf{d_m}} \xleftarrow{\$} \{0,1\}^{3\lambda}$

For $0 \le w \le \mathsf{d_m}$ sample random $\boldsymbol{r}_w$

For i from $\mathsf{d_m} - 1$ to $\ell + 1$ do ▷ Generate "ghost" hash inputs for levels below pth

  $v'^{(i)} \xleftarrow{\$} \{0,1\}^{\lambda}$; $z_i := (h'^{(i+1)}, h'^{(i+1)}, v'^{(i)})$

$(\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \{\mathsf{lb}_{i,b}^{(\mathsf{d_m})}\}) \xleftarrow{\$} \mathsf{Garb}(\mathsf{F}[\mathsf{id}, \mathsf{id}', r, r'])$; $\{\mathsf{lb}_i^{(\mathsf{d_m})}\} := \{\mathsf{lb}_{i,z_{\mathsf{d_m}}[i]}^{(\mathsf{d_m})}\}$

For w from $\mathsf{d_m} - 1$ to p do $(\widetilde{\mathsf{C}}_w, \{\mathsf{lb}_{i,b}^{(w)}\}) \xleftarrow{\$} \mathsf{Garb}(\mathsf{V}[\mathsf{pp}, \mathsf{id}, \{\mathsf{lb}_{i,b}^{(w+1)}\}, \boldsymbol{r}_w, r, \mathsf{id}', r', \mathsf{pad}])$

$\{\mathsf{lb}_i^{(p)}\} := \{\mathsf{lb}_{i,z_p[i]}^{(p)}\}$ ▷ Final set of honest labels

If $p - 1 \ge \ell$ then ▷ If circuits at, or below, pth leaf level are simulated

  If $p - 1 > \ell$ then for $p - 1$ to $\ell + 1$ do ▷ Below pth leaf level

    $\{\mathsf{cth}_{i,b}^{(w+1)}\} \leftarrow \mathsf{HEnc}(\mathsf{pp}, h'^{(w+1)}, \{\mathsf{lb}_{i,b}^{(w+1)}\}; \boldsymbol{r}_w)$; $(\widetilde{\mathsf{C}}_w, \{\mathsf{lb}_i^{(w)}\}) \xleftarrow{\$} \mathsf{Sim}(\mathsf{V}, \{\mathsf{cth}_{i,b}^{(w+1)}\})$

  $(\widetilde{\mathsf{C}}_\ell, \{\mathsf{lb}_i^{(\ell)}\}) \xleftarrow{\$} \mathsf{Sim}(\mathsf{V}, \{\mathsf{id}', r', \mathsf{pad}\})$ ▷ At pth leaf level

  For w from $\ell - 1$ to 0 do ▷ From interior pth nodes to root

    $\{\mathsf{cth}_{i,b}^{(w+1)}\} \leftarrow \mathsf{HEnc}(\mathsf{pp}, h_1^{(w+1)}, \{\mathsf{lb}_i^{(w+1)}\}; \boldsymbol{r}_w)$; $(\widetilde{\mathsf{C}}_w, \{\mathsf{lb}_i^{(w)}\}) \xleftarrow{\$} \mathsf{Sim}(\mathsf{V}, \{\mathsf{cth}_{i,b}^{(w+1)}\})$

Else ▷ If all circuits at, and below, pth leaf level are honest

  For w from $p - 1$ to 0 do

    $\{\mathsf{cth}_{i,b}^{(w+1)}\} \leftarrow \mathsf{HEnc}(\mathsf{pp}, h_1^{(w+1)}, \{\mathsf{lb}_i^{(w+1)}\}; \boldsymbol{r}_w)$; $(\widetilde{\mathsf{C}}_w, \{\mathsf{lb}_i^{(w)}\}) \xleftarrow{\$} \mathsf{Sim}(\mathsf{V}, \{\mathsf{cth}_{i,b}^{(w+1)}\})$

$\{\mathsf{cth}_{i,b}^{(0)}\} \xleftarrow{\$} \mathsf{HEnc}(\mathsf{pp}, \mathsf{h_{root}}, \{\mathsf{lb}_i^{(0)}\})$

Return $\mathsf{m}_2 := (\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_0, \{\mathsf{cth}_{i,b}^{(0)}\}, r)$

Figure 4.5: **$\mathbf{Hyb}_{1.p}$ for $0 \le p \le \mathsf{d_m}$. The last $p+1$ circuits in $\mathbf{Hyb}_{1.p}$ are generated honestly and the remainder are simulated. See Lemma 4.**

same after constructing circuit $\widetilde{\mathsf{C}}_{\mathsf{d_m}}$ and its labels. So, either hybrid can be simulated by knowing r, the induced path pth, and the pair $(\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \{\mathsf{lb}_i^{(\mathsf{d_m})}\})$. For ease of notation let $(\widetilde{\mathsf{C}}, \{\mathsf{lb}_i\})$ and $(\widetilde{\mathsf{C}}', \{\mathsf{lb}'_i\})$ denote the distribution of the tuple $(\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \{\mathsf{lb}_i^{(\mathsf{d_m})}\})$ in $\mathbf{Hyb}_1$ and $\mathbf{Hyb}_{1.0}$, respectively. We have $(\widetilde{\mathsf{C}}, \{\mathsf{lb}_i\}) \xleftarrow{\$} \mathsf{Sim}(\mathsf{F}, \{\mathsf{id}', r'\})$ for random $\mathsf{id}', r' \xleftarrow{\$} \{0,1\}^{\lambda}$. In $\mathbf{Hyb}_{1.0}$, letting $\mathsf{C}_{\mathsf{d_m}} := \mathsf{F}[\mathsf{id}, \mathsf{id}', r, r']$ for random r, we have

$$(\widetilde{\mathsf{C}}', \{\mathsf{lb}_{i,b}\}) \xleftarrow{\$} \mathsf{Garb}(\mathsf{C}_{\mathsf{d_m}})$$
$$\{\mathsf{lb}'_i\} := \{\mathsf{lb}_{i,z_{\mathsf{d_m}}[i]}\},$$

where $z_{\mathsf{d_m}} \xleftarrow{\$} \{0,1\}^{3\lambda}$ if $\ell < \mathsf{d_m}$ and $z_{\mathsf{d_m}} := (\mathsf{Val}[v_1^{(\mathsf{d_m})}]^{\times\lambda}, x_1, x'_1)$ otherwise, where $\mathsf{Val}[v_1^{(\mathsf{d_m})}]^{\times\lambda} =$

0^λ. By simulation security of garbled circuits

$$(\widetilde{\mathsf{C}}', \{\mathsf{lb}'_i\}) \overset{c}{\equiv} \mathsf{Sim}(\mathsf{F}, \mathsf{C}_{\mathsf{d_m}}(z_{\mathsf{d_m}})) \overset{c}{\equiv} \mathsf{Sim}(\mathsf{F}, \{\mathsf{id}', r'\}).$$

If $\ell < \mathsf{d_m}$ and $z_{\mathsf{d_m}}$ is random, then $\mathsf{C}_{\mathsf{d_m}}(z_{\mathsf{d_m}}) = \{\mathsf{id}, r\}$ with probability only $2^{-\lambda}$. If $\ell = \mathsf{d_m}$ and $z_{\mathsf{d_m}} := (0^\lambda, x_1, x'_1)$ then $\mathsf{C}_{\mathsf{d_m}}(z_{\mathsf{d_m}}) = \{\mathsf{id}', r'\}$ with probability 1. Thus, $(r, \mathsf{pth}, \widetilde{\mathsf{C}}, \{\mathsf{lb}_i\}) \overset{c}{\equiv} (r, \mathsf{pth}, \widetilde{\mathsf{C}}', \{\mathsf{lb}'_i\})$, proving $\mathbf{Hyb}_1 \overset{c}{\equiv} \mathbf{Hyb}_{1.0}$, and completing the proof of Lemma 6. □

Lemma 7. *For all $p \in \{0, \ldots, \mathsf{d_m} - 1\}$, hybrids $\mathbf{Hyb}_{1.p}$ and $\mathbf{Hyb}_{1.p+1}$ are computationally indistinguishable.*

Proof. First, consider the circuits created in either hybrid:

$$\mathbf{Hyb}_{1.p} : \overbrace{\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_{p+1}, \widetilde{\mathsf{C}}_p}^{\text{Garb}}, \overbrace{\widetilde{\mathsf{C}}_{p-1}, \ldots, \widetilde{\mathsf{C}}_0}^{\text{Sim}}$$
$$\mathbf{Hyb}_{1.p+1} : \underbrace{\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_{p+1}}_{\text{Garb}}, \underbrace{\widetilde{\mathsf{C}}_p, \widetilde{\mathsf{C}}_{p-1}, \ldots, \widetilde{\mathsf{C}}_0}_{\text{Sim}}$$

It is clear that $(\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \{\mathsf{lb}^{(\mathsf{d_m})}_{i,b}\}, \ldots, \widetilde{\mathsf{C}}_{p+1}, \{\mathsf{lb}^{(p+1)}_{i,b}\})$ are formed the same way in both hybrids. The two hybrids differ only in the distribution of $(\widetilde{\mathsf{C}}_p, \{\mathsf{lb}^{(p)}_i\})$; it is generated honestly in $\mathbf{Hyb}_{1.p}$ and simulated in $\mathbf{Hyb}_{1.p+1}$.

There are three possible ways $(\widetilde{\mathsf{C}}_p, \{\mathsf{lb}^{(p)}_i\})$ can be simulated in $\mathbf{Hyb}_{1.p+1}$ depending on the value of p relative to ℓ (see Fig. 4.4, but note that the figure illustrates $\mathbf{Hyb}_{1.p}$, not $\mathbf{Hyb}_{1.p+1}$). First, if $p < \ell$ holds, then $(\widetilde{\mathsf{C}}_p, \{\mathsf{lb}^{(p)}_i\})$ is simulated using a hash encryption of $\{\mathsf{lb}^{(p+1)}_i\}$ with z_{p+1}. If $p = \ell$, then $(\widetilde{\mathsf{C}}_p, \{\mathsf{lb}^{(p)}_i\})$ is simulated using random output since $\mathsf{BP}(\mathsf{id}) = 0$. Finally, if $p > \ell$ holds, then $(\widetilde{\mathsf{C}}_p, \{\mathsf{lb}^{(p)}_i\})$ is simulated using a hash encryption of $\{\mathsf{lb}^{(p+1)}_i\}$ using "ghost" value z_{p+1}. We will prove that in each of these three possible combinations it holds that $(\widetilde{\mathsf{C}}_p, \{\mathsf{lb}^{(p)}_i\})_{\mathbf{Hyb}_{1.p}} \overset{c}{\equiv} (\widetilde{\mathsf{C}}_p, \{\mathsf{lb}^{(p)}_i\})_{\mathbf{Hyb}_{1.p+1}}$.

1. If $p < \ell$:

$$\mathbf{Hyb}_{1.p} : \begin{cases} (\widetilde{\mathsf{C}}_p, \{\mathsf{lb}^{(p)}_{i,b}\}) \overset{\$}{\leftarrow} \mathsf{Garb}(\mathsf{V}[\mathsf{pp}, \mathsf{id}, \{\mathsf{lb}^{(p+1)}_{i,b}\}, \boldsymbol{r}_p, r, \mathsf{id}', r', \mathsf{pad}]) \\ \{\mathsf{lb}^{(p)}_i\} := \{\mathsf{lb}^{(p)}_{i,z_p[i]}\} \quad \text{where } z_p = (h^{(p+1)}_1, h^{(p+1)}_2, v^{(p)}_1) \end{cases}$$
$$\mathbf{Hyb}_{1.p+1} : \begin{cases} \{\mathsf{cth}^{(p+1)}_{i,b}\} \leftarrow \mathsf{HEnc}(\mathsf{pp}, h^{(p+1)}_1, \{\mathsf{lb}^{(p+1)}_i\}; \boldsymbol{r}_p) \\ (\widetilde{\mathsf{C}}_p, \{\mathsf{lb}^{(p)}_i\}) \overset{\$}{\leftarrow} \mathsf{Sim}(\mathsf{V}, \{\mathsf{cth}^{(p+1)}_{i,b}\}) \end{cases} \tag{4.9}$$

By simulation security of garbled circuits,

$$\begin{aligned}(\widetilde{\mathsf{C}}_p, \{\mathsf{lb}_i^{(p)}\})_{\mathbf{Hyb}_{1.p}} &\overset{c}{\equiv} \mathsf{Sim}(\mathsf{V}, \mathsf{C}_p(z_p)) \\ &\overset{c}{\equiv} \mathsf{Sim}(\mathsf{V}, \mathsf{HEnc}(\mathsf{pp}, h_1^{(p+1)}, \{\mathsf{lb}_{i,b}^{(p+1)}\}; \boldsymbol{r}_p)). \end{aligned} \tag{4.10}$$

The use of $h_1^{(p+1)}$ in Eq. 4.10 is due to the assumption that the path induced by id has the leftmost node as its terminal node. So by definition of C_p, its hardwired labels $\{\mathsf{lb}_{i,b}^{(p+1)}\}$ will be encrypted under $h_1^{(p+1)}$. Eq. 4.10 is identical to the RHS of Eq. 4.9, and thus when $p > \ell$ we have $(\widetilde{\mathsf{C}}_p, \{\mathsf{lb}_i^{(p)}\})_{\mathbf{Hyb}_{1.p}} \overset{c}{\equiv} (\widetilde{\mathsf{C}}_p, \{\mathsf{lb}_i^{(p)}\})_{\mathbf{Hyb}_{1.p+1}}$.

2. If $p = \ell$:

$$\begin{aligned}\mathbf{Hyb}_{1.p} &: \begin{cases}(\widetilde{\mathsf{C}}_p, \{\mathsf{lb}_{i,b}^{(p)}\}) \overset{\$}{\leftarrow} \mathsf{Garb}(\mathsf{V}[\mathsf{pp}, \mathsf{id}, \{\mathsf{lb}_{i,b}^{(p+1)}\}, \boldsymbol{r}_p, r, \mathsf{id}', r', \mathsf{pad}]) \\ \{\mathsf{lb}_i^{(p)}\} := \{\mathsf{lb}_{i,z_p[i]}^{(p)}\} \quad \text{where } z_p = (0^\lambda, x_1, x_1')\end{cases} \\ \mathbf{Hyb}_{1.p+1} &: \begin{cases}(\widetilde{\mathsf{C}}_p, \{\mathsf{lb}_i^{(p)}\}) \overset{\$}{\leftarrow} \mathsf{Sim}(\mathsf{V}, \{\mathsf{id}', r', \mathsf{pad}\}) \\ \text{where } \mathsf{id}', r' \overset{\$}{\leftarrow} \{0,1\}^\lambda ; \mathsf{pad} \overset{\$}{\leftarrow} \{0,1\}^{2(n-1)}\end{cases}\end{aligned} \tag{4.11}$$

By simulation security of garbled circuits,

$$\begin{aligned}(\widetilde{\mathsf{C}}_p, \{\mathsf{lb}_i^{(p)}\})_{\mathbf{Hyb}_{1.p}} &\overset{c}{\equiv} \mathsf{Sim}(\mathsf{V}, \mathsf{C}_p(z_p)) \\ &\overset{c}{\equiv} \mathsf{Sim}(\mathsf{V}, \{\mathsf{id}', r', \mathsf{pad}\}) \,. \end{aligned} \tag{4.12}$$

When $p = \ell$, $z_p = (0^\lambda, x_1, x_1')$ which causes $\mathsf{C}_p(z_p)$ to output a random ID and signal string. So, Equation 4.12 is identical to the first line of Eq. 4.11. Thus if $p = \ell$, we have $(\widetilde{\mathsf{C}}_p, \{\mathsf{lb}_i^{(p)}\})_{\mathbf{Hyb}_{1.p}} \overset{c}{\equiv} (\widetilde{\mathsf{C}}_p, \{\mathsf{lb}_i^{(p)}\})_{\mathbf{Hyb}_{1.p+1}}$.

3. If $p > \ell$:

$$\begin{aligned}\mathbf{Hyb}_{1.p} &: \begin{cases}(\widetilde{\mathsf{C}}_p, \{\mathsf{lb}_{i,b}^{(p)}\}) \overset{\$}{\leftarrow} \mathsf{Garb}(\mathsf{V}[\mathsf{pp}, \mathsf{id}, \{\mathsf{lb}_{i,b}^{(p+1)}\}, \boldsymbol{r}_p, r, \mathsf{id}', r', \mathsf{pad}]) \\ \{\mathsf{lb}_i^{(p)}\} := \{\mathsf{lb}_{i,z_p[i]}^{(p)}\} \quad \text{where } z_p = (h'^{(p+1)}, h'^{(p+1)}, v'^{(p)})\end{cases} \\ \mathbf{Hyb}_{1.p+1} &: \begin{cases}\{\mathsf{cth}_{i,b}^{(p+1)}\} \leftarrow \mathsf{HEnc}(\mathsf{pp}, h'^{(p+1)}, \{\mathsf{lb}_i^{(p+1)}\}; \boldsymbol{r}_p) \\ (\widetilde{\mathsf{C}}_p, \{\mathsf{lb}_i^{(p)}\}) \overset{\$}{\leftarrow} \mathsf{Sim}(\mathsf{V}, \{\mathsf{cth}_{i,b}^{(p+1)}\}) \\ \text{where } h'^{(p+1)} \leftarrow \mathsf{Hash}(\mathsf{pp}, z_{p+1}) \text{ is pseudorandom}\end{cases}\end{aligned} \tag{4.13}$$

Consider evaluating $\widetilde{\mathsf{C}}_p$ on labels $\{\mathsf{lb}^{(p)}_{i,z_p[i]}\}$ as in $\mathbf{Hyb}_{1.p}$:

$$\begin{aligned}\mathsf{Eval}(\widetilde{\mathsf{C}}_p, \{\mathsf{lb}^{(p)}_{i,z_p[i]}\}) &= \mathsf{V}[\mathsf{pp}, \mathsf{id}, \{\mathsf{lb}^{(p+1)}_{i,b}\}, \boldsymbol{r}_p, r, \mathsf{id}', r', \mathsf{pad}](h'^{(p+1)}, h'^{(p+1)}, v'^{(p)}) \\ &= \begin{cases} \mathsf{HEnc}(\mathsf{pp}, h'^{(p+1)}, \{\mathsf{lb}^{(p+1)}_{i,b}\}; \boldsymbol{r}_p) & \text{if } \mathsf{id}[v'_p] = 0 \\ \mathsf{HEnc}(\mathsf{pp}, h'^{(p+1)}, \{\mathsf{lb}^{(p+1)}_{i,b}\}; \boldsymbol{r}_p) & \text{otherwise} \end{cases} \\ &= \mathsf{HEnc}(\mathsf{pp}, h'^{(p+1)}, \{\mathsf{lb}^{(p+1)}_{i,b}\}; \boldsymbol{r}_p)\,. \end{aligned} \tag{4.14}$$

Equation 4.14 is identical to the RHS of Eq. 4.13 (first), up to the labels $\{\mathsf{lb}^{(p+1)}_i\}$ in Eq. 4.13 vs. $\{\mathsf{lb}^{(p+1)}_{i,b}\}$ in Eq. 4.14. By simulation security, the labels $\{\mathsf{lb}^{(p+1)}_i\}$ in Eq. 4.13 are computationally indistinguishable from labels $\{\mathsf{lb}^{(p+1)}_{i,z_{p+1}[i]}\}$. Thus $\{\mathsf{lb}^{(p+1)}_{i,z_{p+1}[i]}\}_{\mathbf{Hyb}_{1.p}} \overset{c}{\equiv} \{\mathsf{lb}^{(p+1)}_i\}_{\mathbf{Hyb}_{1.p+1}}$. By HE semantic security, $\mathsf{HEnc}(\mathsf{pp}, h'^{(p+1)}, \{\mathsf{lb}^{(p+1)}_{i,b}\}; \boldsymbol{r}_p) \overset{c}{\equiv} \mathsf{HEnc}(\mathsf{pp}, h'^{(p+1)}, \{\mathsf{lb}^{(p+1)}_i\}; \boldsymbol{r}_p)$, and hence $(\widetilde{\mathsf{C}}_p, \{\mathsf{lb}^{(p)}_i\})_{\mathbf{Hyb}_{1.p}} \overset{c}{\equiv} (\widetilde{\mathsf{C}}_p, \{\mathsf{lb}^{(p)}_i\})_{\mathbf{Hyb}_{1.p+1}}$ when $p > \ell$, which completes the proof of Lemma 7.

□

4.3.2 Proof of Lemma 5

Lemma 5 states that $\mathbf{Hyb}_1 \overset{c}{\equiv} \mathbf{Hyb}_2$. So, we must show that $\mathsf{m}_2 := (\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_0, \{\mathsf{cth}^{(0)}_{i,b}\}, r)$, as sampled in $\mathbf{Hyb}_1$, is computationally indistinguishable from random. Recall that in $\mathbf{Hyb}_1$, all circuits are simulated. We will argue that each element of m_2 is pseudorandom, then that the joint distribution is pseudorandom.

First consider the circuit $\widetilde{\mathsf{C}}_{\mathsf{d_m}}$. It is formed as $(\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \{\mathsf{lb}^{(\mathsf{d_m})}_i\}) \overset{\$}{\leftarrow} \mathsf{Sim}(\mathsf{F}, \{\mathsf{id}', r'\})$ where $\mathsf{id}', r' \overset{\$}{\leftarrow} \{0,1\}^\lambda$. Since the inputs id', r' are random, by anonymous security of garbled circuits the distribution $(\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \{\mathsf{lb}^{(\mathsf{d_m})}_i\})$ is pseudorandom.

For w from $\mathsf{d_m} - 1$ to $\ell + 1$ the circuits are formed as $(\widetilde{\mathsf{C}}_w, \{\mathsf{lb}^{(w)}_i\}) \overset{\$}{\leftarrow} \mathsf{Sim}(\mathsf{V}, \{\mathsf{cth}^{(w+1)}_{i,b}\})$ where $\{\mathsf{cth}^{(w+1)}_{i,b}\} \overset{\$}{\leftarrow} \mathsf{HEnc}(\mathsf{pp}, h'^{(w+1)}, \{\mathsf{lb}^{(w+1)}_i\})$. $\{\mathsf{cth}^{(w+1)}_{i,b}\}$ is pseudorandom by anonymous semantic security of HE, and so by anonymous security of GS, $(\widetilde{\mathsf{C}}_w, \{\mathsf{lb}^{(w)}_i\})$ is also pseudorandom.

For $w = \ell$ we have $(\widetilde{\mathsf{C}}_\ell, \{\mathsf{lb}^{(\ell)}_i\}) \overset{\$}{\leftarrow} \mathsf{Sim}(\mathsf{V}, \{\mathsf{id}', r', \mathsf{pad}\})$ where $\mathsf{id}', r' \overset{\$}{\leftarrow} \{0,1\}^\lambda$, $\mathsf{pad} \overset{\$}{\leftarrow} \{0,1\}^{2(n-1)}$, so again by anonymous security of garbled circuits, the distribution $(\widetilde{\mathsf{C}}_\ell, \{\mathsf{lb}^{(\ell)}_i\})$ is pseudorandom.

For w from $\ell-1$ to 0 we have $\{\mathsf{cth}_{i,b}^{(w+1)}\} \xleftarrow{\$} \mathsf{HEnc}(\mathsf{pp}, h_1^{(w+1)}, \{\mathsf{lb}_i^{(w+1)}\})$ and $(\widetilde{\mathsf{C}}_w, \{\mathsf{lb}_i^{(w)}\}) \xleftarrow{\$} \mathsf{Sim}(\mathsf{V}, \{\mathsf{cth}_{i,b}^{(w+1)}\})$. Where, again, the use of $h_1^{(w+1)}$ in HEnc is from the assumption on pth. For all w from $\ell-1$ to 0, $\{\mathsf{cth}_{i,b}^{(w+1)}\}$ is pseudorandom by anonymous semantic security of HE, and thus by anonymous security of GS, $(\widetilde{\mathsf{C}}_w, \{\mathsf{lb}_i^{(w)}\})$ is also pseudorandom.

Next in m_2 is the ciphertext, which in $\mathbf{Hyb}_1$ is formed as $\{\mathsf{cth}_{i,b}^{(0)}\} \xleftarrow{\$} \mathsf{HEnc}(\mathsf{pp}, \mathsf{h}_{\mathsf{root}}, \{\mathsf{lb}_i^{(0)}\})$. $\{\mathsf{cth}_{i,b}^{(0)}\}$ is pseudorandom by anonymous semantic security of HE. The final element of m_2 is the signal string r, which is sampled uniformly at random.

Now that we have shown each element of m_2 with simulated circuits is pseudorandom, it remains to show that the joint distribution is also pseudorandom. We will do so by showing each of the following hybrid hops are computationally indistinguishable.

$$\mathbf{Hyb}_1: \quad \mathsf{m}_2 = (\overbrace{\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_0}^{\mathsf{Sim}}, \quad \overbrace{\{\mathsf{cth}_{i,b}^{(0)}\}}^{\mathsf{HEnc}(\mathsf{pp},\mathsf{h}_{\mathsf{root}},\{\mathsf{lb}_i^{(0)}\})}, \overbrace{r}^{\$}) \tag{4.15}$$

$$\mathsf{m}_2 = (\overbrace{\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_0}^{\mathsf{Sim}}, \quad \overbrace{\{\mathsf{cth}_{i,b}^{(0)}\}}^{\mathsf{HEnc}(\mathsf{pp},\mathsf{h}_{\mathsf{root}},\{\mathsf{lb}_i^{(0)}\})}, \overbrace{r}^{\$'}) \tag{4.16}$$

$$\mathsf{m}_2 = (\overbrace{\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_0}^{\mathsf{Sim}}, \quad \overbrace{\{\mathsf{cth}_{i,b}^{(0)}\}}^{\$}, \overbrace{r}^{\$'}) \tag{4.17}$$

$$\mathsf{m}_2 = (\overbrace{\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_1}^{\mathsf{Sim}}, \overbrace{\widetilde{\mathsf{C}}_0}^{\$}, \quad \overbrace{\{\mathsf{cth}_{i,b}^{(0)}\}}^{\$}, \overbrace{r}^{\$'}) \tag{4.18}$$

$$\mathsf{m}_2 = (\overbrace{\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_{i+1}}^{\mathsf{Sim}}, \overbrace{\widetilde{\mathsf{C}}_i, \ldots, \widetilde{\mathsf{C}}_0}^{\$}, \quad \overbrace{\{\mathsf{cth}_{i,b}^{(0)}\}}^{\$}, \overbrace{r}^{\$'}) \tag{4.19}$$

$$\vdots$$

$$\mathbf{Hyb}_2: \quad \mathsf{m}_2 = (\overbrace{\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_0}^{\$}, \quad \overbrace{\{\mathsf{cth}_{i,b}^{(0)}\}}^{\$}, \overbrace{r}^{\$'}) \tag{4.20}$$

Eq. 4.15 to Eq. 4.16: In Eq. 4.15, $\widetilde{\mathsf{C}}_{\mathsf{d_m}}, \ldots, \widetilde{\mathsf{C}}_0$ are simulations of circuits with r hardcoded, while in Eq. 4.16, r is randomly sampled independently. Note that because of the assumption $\mathsf{BP}(\mathsf{id}) = 0$, no evaluation of the circuits using $\{\mathsf{cth}_{i,b}^{(0)}\}$ and $\mathsf{DecPath}$ will result in the output of the signal string r. So, the simulated circuits do not actually have r hardcoded. Hence, the signal string r in $\mathbf{Hyb}_1$ will be independent of the other elements of m_2 (the simulated circuits and the ciphertext $\{\mathsf{cth}_{i,b}^{(0)}\}$). So the change in Eq. 4.16 to an independently random string is computationally undetectable.

Eq. 4.16 to Eq. 4.17: Here, the ciphertext changes from an encryption of the simulated labels for $\widetilde{\mathsf{C}}_1$ to independently random. As mentioned above, by the assumption $\mathsf{BP}(\mathsf{id}) = 0$, the evaluation of $\mathsf{DecPath}$ will only ever output $(\mathsf{id}', r') \neq (\mathsf{id}, r)$. So changing the ciphertext to random does not affect the evaluation of the circuits. On the other hand, the simulated circuits contain no information about $\{\mathsf{cth}_{i,b}^{(0)}\}$ that could help in distinguishing between Eq. 4.16 and Eq. 4.17.

Eq. 4.17 to Eq. 4.18: Before switching to random, $\widetilde{\mathsf{C}}_0$ depends on the hash encryption of labels of $\widetilde{\mathsf{C}}_1$. But since the labels to evaluate $\widetilde{\mathsf{C}}_0$ were lost in Eq. 4.17 when the ciphertext was switched to random, $\widetilde{\mathsf{C}}_0$ can no longer be evaluated. Hence $\widetilde{\mathsf{C}}_0$'s dependence on $\widetilde{\mathsf{C}}_1$ is hidden by semantic security of hash encryption.

Eq. 4.18 to Eq. 4.19 for $i = 1$ and Eq. 4.19 for i increasing to d_m: Each of these hops is indistinguishable due to a similar argument as the previous hop. The difference is that instead of the ciphertext being switched to random, the circuit which outputs the ciphertext is now switched to random. The result is the same as the previous hop and the change is indistinguishable.

Hence m_2 is pseudorandom in the view of R, proving $\mathbf{Hyb}_1 \stackrel{c}{\equiv} \mathbf{Hyb}_2$ and completing the proof. □

Remark 1. *In the proofs above, we assumed that the path induced by evaluating* $\mathsf{BP}(\mathsf{id})$ *always travelled to the left child. In the general case, the path in Eq. 4.8 ending in* $v_1^{(\ell)}$ *just needs to be changed to the path induced by* $\mathsf{BP}(\mathsf{id})$ *ending in the appropriate leaf* $u \in T$*. The proofs should then be updated accordingly.*

Chapter 5

Applications

Construction 1 for BP-2PC can be used to realize multiple functionalities by reducing the desired functionality to an instance of BP-2PC. One step of the reductions involves constructing a branching program based on a set of bit strings.

At a high level, SetBP (Fig. 5.1) creates a branching program for a set of elements (each of length λ) $S := \{x_1, \ldots, x_m\}$ in three main steps. For concreteness, suppose the goal is to use this BP for a private set intersection.

First, for every prefix $a \in \{\epsilon\} \cup \{0,1\} \cup \{0,1\}^2 \cup \cdots \cup \{0,1\}^\lambda$ of the elements in S, a node, v_a, is added to the set of nodes V. If $a \in S$, then the value encoded in v_a is set to 1. This is an 'accept' leaf node. If $|a| < \lambda$, then the value encoded in v_a is set to $|a| + 1$. When the BP is being evaluated on some input, this will indicate the bit following prefix a. Next, edges are created between levels of the BP. For $|a| < \lambda$, if for $b \in \{0,1\}$, node $v_{a\|b}$ exists in V, then an edge labeled with bit b is created from v_a to $v_{a\|b}$. For $b \in \{0,1\}$, if $v_{a\|b} \notin V$, then node $v_{a\|b}$ is added to V with an encoded bit 0. This is a 'reject' leaf. Then a b-labelled edge is added from v_a to $v_{a\|b}$. Lastly, the BP is pruned. If two sibling leaves are both encoded with the same value, they are deleted and their parent becomes a leaf encoding that same value.

The definition below generalizes this concept by allowing us to capture both PSI and PSU via an indicator bit b_{pth}. In the description above, b_{pth} is set to 1 for the PSI setting. For PSU we set $b_{\mathsf{pth}} = 0$.

Construction 2 (Set to branching program)**.** *Figure 5.1 defines a procedure to create a branching program from an input set S.* $\mathsf{SetBP}(S, b_{\mathsf{pth}})$ *takes as input a set $S := \{x_1, \ldots, x_m\}$ of m strings, all of length λ and a bit b_{pth} and outputs a tuple (V, E, T, Val)*

defining a branching program. The output BP is such that if $x \in S$*, then* $\mathsf{BP}(x) = b_{\mathsf{pth}}$*, and if* $x \notin S$*, then* $\mathsf{BP}(x) = 1 - b_{\mathsf{pth}}$*.*

Procedure SetBP *runs in time* $O(\lambda|S|)$*. In particular, when* $|S| = \mathsf{poly}(\lambda)$*,* SetBP *generates the BP in time* $O(\mathsf{poly}(\lambda))$*. The output BP has depth* $d \leq \lambda + 1$ *and the number of nodes is* $2d + 1 \leq |V| \leq 2^{d+1} - 1$*. Evaluation of* $\mathsf{BP}(x)$ *for arbitrary* $x \in \{0,1\}^\lambda$ *takes time* $O(\lambda)$*.*

BP evaluation runtime: Recall the BP evaluation algorithm in Fig. 3.1. Each loop iteration of the evaluation makes progress by moving down the tree one level. The number of iterations is at most the tree depth, which is at most $\lambda+1$ for the BP created in Fig. 5.1. Each iteration takes constant time, so evaluation of $\mathsf{BP}(x)$ for arbitrary $x \in \{0,1\}^\lambda$ takes time $O(\lambda)$.

5.1 Private Set Intersection (PSI)

Assume a sender party has a singleton set $S_{\mathsf{S}} = \{\mathsf{id}\}$ where $\mathsf{id} \in \{0,1\}^\lambda$ and a receiver has a set $S_{\mathsf{R}} \subset \{0,1\}^\lambda$ such that $|S_{\mathsf{R}}|$ is $\mathsf{poly}(\lambda)$. In this setting, we can define PSI as follows.

Definition 8 (Private set union (PSI) functionality with $|S_{\mathsf{S}}| = 1$)**.** *Let* Π *be a two-party communication protocol. Let* R *be the receiver holding set* $S_{\mathsf{R}} \subset \{0,1\}^\lambda$ *and let* S *be the sender holding singleton set* $S_{\mathsf{S}} = \{\mathsf{id}\}$*, with* $\mathsf{id} \in \{0,1\}^\lambda$*.* Π *is a PSI protocol if the following hold after execution of the protocol.*

- ***Correctness:*** *R learns* $S_{\mathsf{R}} \cap \{\mathsf{id}\}$ *if and only if* $\mathsf{id} \in S_{\mathsf{R}}$*.*
- ***Receiver security:*** Π *achieves receiver security if for all* $\mathsf{id} \in \{0,1\}^\lambda$*, and all pairs* $S_{\mathsf{R}0}, S_{\mathsf{R}1} \subset \{0,1\}^\lambda$ *we have that* $\mathsf{view}^\Pi_S(S_{\mathsf{R}0}, \mathsf{id}, \lambda) \approx \mathsf{view}^\Pi_S(S_{\mathsf{R}1}, \mathsf{id}, \lambda)$*. If the distributions are computationally (resp., statistically) indistinguishable then we have computational (resp., statistical) security.*
- ***Sender security:*** Π *achieves security for the sender if for all* $\lambda \in \mathbb{N}$*,* $S_{\mathsf{R}} \subset \{0,1\}^\lambda$*, and all pairs* $\mathsf{id}_0, \mathsf{id}_1 \in \{0,1\}^\lambda \setminus S_{\mathsf{R}}$ *we have that* $\mathsf{view}^\Pi_R(S_{\mathsf{R}}, \mathsf{id}_0, \lambda) \approx \mathsf{view}^\Pi_R(S_{\mathsf{R}}, \mathsf{id}_1, \lambda)$*. If the distributions are computationally (resp., statistically) indistinguishable then we have computational (resp., statistical) security.*
- ***Security against outsiders:*** Π *is secure against outsiders if for all outside parties* $P \notin \{S, R\}$ *and all pairs of sender/receiver inputs* $(S_{\mathsf{R}0}, \mathsf{id}_0)$ *and* $(S_{\mathsf{R}1}, \mathsf{id}_1)$*, we have that* $\mathsf{tr}^\Pi(\lambda, S_{\mathsf{R}0}, \mathsf{id}_0) \approx \mathsf{tr}^\Pi(\lambda, S_{\mathsf{R}1}, \mathsf{id}_1)$*.*

Procedure $\mathsf{SetBP}(S, b_{\mathsf{pth}})$**:**

$\{x_1, \ldots, x_m\} \leftarrow S$; $\lambda \leftarrow |x_1|$; $V, E, T \leftarrow \emptyset$

$V \leftarrow V \cup \{v_\epsilon\}$; $\mathsf{Val}[v_\epsilon] \leftarrow 1$ ▷ set root node

For $1 \leq i \leq \lambda$ do ▷ add a node for every prefix of length i in S

  For $1 \leq j \leq m$ do

    $a \leftarrow x_j[1..i]$; $V \leftarrow V \cup \{v_a\}$

    If $|a| = \lambda$ then $\mathsf{Val}[v_a] \leftarrow b_{\mathsf{pth}}$; $T \leftarrow T \cup \{v_a\}$ ▷ accept leaves

    Else $\mathsf{Val}[v_a] \leftarrow |a| + 1$

For all $v_a \in V$ s.t. $|v_a| < \lambda$ do ▷ adding edges from v_a to children

  For $b \in \{0, 1\}$ do

    If $\exists\ v_{a\|b} \in V$ then $E \leftarrow E \cup \{(v_a, v_{a\|b}, b)\}$

    Else

      $V \leftarrow V \cup \{v_{a\|b}\}$; $\mathsf{Val}[v_{a\|b}] \leftarrow 1 - b_{\mathsf{pth}}$ ▷ reject leaves

      $E \leftarrow E \cup \{(v_a, v_{a\|b}, b)\}$; $T \leftarrow T \cup \{v_{a\|b}\}$

▷ Pruning: if a node has 2 leaf children with value b_{pth}, delete the children and change parent value to b_{pth}.

While $\exists\ v_a \in V$ s.t. $v_{a\|b} \in T \wedge \mathsf{Val}[v_{a\|b}] = b_{\mathsf{pth}}$ for $b \in \{0, 1\}$ do

  $\mathsf{Val}[v_a] \leftarrow b_{\mathsf{pth}}$; $T \leftarrow T \cup \{v_a\}$; $T \leftarrow T \setminus \{v_{a\|0}, v_{a\|1}\}$

  $V \leftarrow V \setminus \{v_{a\|0}, v_{a\|1}\}$; $E \leftarrow E \setminus \{(v_a, v_{a\|b}, b) \mid b \in \{0, 1\}\}$

Return (V, E, T, Val)

Figure 5.1: **Procedure for constructing a BP from a set of m λ-bit strings. See Construction 2. Based on a description in [CGH+21].**

The PSI functionality can be achieved by casting it as an instance of BP-2PC:

1. R runs $\mathsf{SetBP}(S_{\mathsf{R}}, 1)$ (Fig. 5.1) to generate a branching program $\mathrm{BP}_{\mathrm{psi}}$ such that $\mathsf{BP}_{\mathrm{psi}}(x) = 1$ if $x \in S_{\mathsf{R}}$ and $\mathsf{BP}_{\mathrm{psi}}(x) = 0$ otherwise.

2. R and S run BP-2PC with inputs $\mathrm{BP}_{\mathrm{psi}}$ and id, respectively. By construction of BP-2PC:

$$\begin{cases} R \text{ learns } \mathsf{id} & \text{if } \mathrm{BP}_{\mathrm{psi}}(\mathsf{id}) = 1 \implies \mathsf{id} \in S_{\mathsf{R}} \\ R \text{ does not learn } \mathsf{id} & \text{if } \mathrm{BP}_{\mathrm{psi}}(\mathsf{id}) = 0 \implies \mathsf{id} \notin S_{\mathsf{R}} \end{cases},$$

which satisfies the PSI correctness condition and security follows from the security of Construction 1 for BP-2PC.

Note that the computation and communication costs of the receiver and sender do not depend on $|S_{\mathsf{R}}|$. Suppose the receiver holds a BP with a polynomial number of root-to-leaf paths that describes a set S_{R} of exponential size. Then, this PSI protocol can run in polynomial time.[1]

5.2 Private Set Union (PSU)

As before, assume the sender has a singleton set $S_{\mathsf{S}} = \{\mathsf{id}\}$ where $\mathsf{id} \in \{0,1\}^\lambda$ and the receiver has a set S_{R}. In this setting, we define PSU as follows.

Definition 9 (PSU functionality with $|S_{\mathsf{S}}| = 1$)**.** *Let Π be a two-party communication protocol. Let R be the receiver holding set $S_{\mathsf{R}} \subset \{0,1\}^\lambda$ and let S be the sender holding singleton set $S_{\mathsf{S}} = \{\mathsf{id}\}$, with $\mathsf{id} \in \{0,1\}^\lambda$. Π is a PSU protocol if the following hold after execution of the protocol.*

- ***Correctness:*** *R learns $S_{\mathsf{R}} \cup \{\mathsf{id}\}$.*
- ***Receiver security:*** *Π achieves receiver security if for all $\mathsf{id} \in \{0,1\}^\lambda$, and all pairs $S_{\mathsf{R}0}, S_{\mathsf{R}1} \subset \{0,1\}^\lambda$ we have that $\mathsf{view}^{\Pi}_{S}(S_{\mathsf{R}0}, \mathsf{id}, \lambda) \approx \mathsf{view}^{\Pi}_{S}(S_{\mathsf{R}1}, \mathsf{id}, \lambda)$. If the distributions are computationally (resp., statistically) indistinguishable then we have computational (resp., statistical) security.*
- ***Sender security:*** *Π achieves security for the sender if for all $S_{\mathsf{R}} \subset \{0,1\}^\lambda$, and all pairs $\mathsf{id}_0, \mathsf{id}_1 \in S_{\mathsf{R}}$ we have that $\mathsf{view}^{\Pi}_{R}(S_{\mathsf{R}}, \mathsf{id}_0, \lambda) \approx \mathsf{view}^{\Pi}_{R}(S_{\mathsf{R}}, \mathsf{id}_1, \lambda)$. If the distributions are computationally (resp., statistically) indistinguishable then we have computational (resp., statistical) security.*
- ***Security against outsiders:*** *Π is secure against outsiders if for all outside parties $P \notin \{S, R\}$ and all pairs of sender/receiver inputs $(S_{\mathsf{R}0}, \mathsf{id}_0)$ and $(S_{\mathsf{R}1}, \mathsf{id}_1)$, we have that $\mathsf{tr}^{\Pi}(\lambda, S_{\mathsf{R}0}, \mathsf{id}_0) \approx \mathsf{tr}^{\Pi}(\lambda, S_{\mathsf{R}1}, \mathsf{id}_1)$.*

The PSU functionality can be achieved by casting it as an instance of BP-2PC:

[1] This assumes R already holds the BP and does not have to build it from their exponential-sized set S_{R}.

1. R runs $\mathsf{SetBP}(S_{\mathsf{R}}, 0)$ (Fig. 5.1) to generate a branching program $\mathrm{BP}_{\mathrm{psu}}$ such that $\mathsf{BP}_{\mathrm{psu}}(x) = 1$ if $x \notin S_{\mathsf{R}}$ and $\mathsf{BP}_{\mathrm{psu}}(x) = 0$ otherwise.

2. R and S run $\mathsf{BP\text{-}2PC}$ with inputs $\mathrm{BP}_{\mathrm{psu}}$ and id, respectively. By construction of $\mathsf{BP\text{-}2PC}$:

$$\begin{cases} R \text{ learns } \mathsf{id} & \text{if } \mathrm{BP}_{\mathrm{psu}}(\mathsf{id}) = 1 \implies \mathsf{id} \notin S_{\mathsf{R}} \\ R \text{ does not learn } \mathsf{id} & \text{if } \mathrm{BP}_{\mathrm{psu}}(\mathsf{id}) = 0 \implies \mathsf{id} \in S_{\mathsf{R}} \end{cases},$$

   which satisfies the PSU correctness condition and security follows from the security of Construction 1 for $\mathsf{BP\text{-}2PC}$.

Note that the computation and communication costs of the receiver and sender do not depend on $|S_{\mathsf{R}}|$. Suppose the receiver holds a BP with a polynomial number of root-to-leaf paths that describes a set S_{R} of exponential size. Then, this PSU protocol can run in polynomial time.[2]

5.3 Wildcards

Definition 10 (Wildcard)**.** *In a bit string a wildcard, denoted by an asterisk $*$, is used in place of a bit to indicate that its position may hold either bit value. In particular, the wildcard character replaces only a single bit, not a string.*

For example, $00* = \{000, 001\}$ and $**0 = \{000, 010, 100, 110\}$.

SetBP in Fig. 5.1 assumes receiver's set S_{R} does not contain strings with wildcards. Fig. 5.2 presents a modified version called SetBP^* which creates a branching program based on a singleton set S_{R} containing a string with wildcard elements. Using SetBP^* instead of SetBP in the constructions for PSI and PSU above allows the receiver's set to contain wildcards.

SetBP^* runs in $O(\lambda)$ time. The resulting BP will contain $2\overline{k} + 1$ nodes, where $\overline{k} \leq \lambda$ is the number of non-wildcard indices. The BP has depth $\overline{k}$, or $\lambda - k$, where k is the number of wildcard indices. Since the depth leaks the number of wildcards in x, the receiver's message m_1 to the sender in Construction 1 contains the maximum depth d_{m}, instead of the true depth.

[2]This assumes R already holds the BP and does not have to build it from their exponential-sized set S_{R}.

Procedure $\mathsf{SetBP}^*(S, b_{\mathsf{pth}})$**:**

$x \leftarrow S$; $\lambda \leftarrow |x|$; $V, E, T \leftarrow \emptyset$

$\overline{\mathsf{WC}} \leftarrow \{i \mid x[i] \neq *\}$; $\overline{k} \leftarrow |\overline{\mathsf{WC}}|$ ▷ ascending ordered set of all non-wildcard indices

If $x[1] \neq *$ then $V \leftarrow V \cup \{v_\epsilon\}$; $\mathsf{Val}[v_\epsilon] \leftarrow 1$ ▷ root node if x doesn't start with *

If $x[1] = *$ then $V \leftarrow V \cup \{v_\epsilon\}$; $\mathsf{Val}[v_\epsilon] \leftarrow \overline{\mathsf{WC}}[1]$ ▷ root node if x starts with *

For $1 \leq i \leq \overline{k}$ do ▷ for every non-wildcard index of x

  $j \leftarrow \overline{\mathsf{WC}}[i]$; $a \leftarrow x[1..j]$; $V \leftarrow V \cup \{v_a\}$

  If $i = \overline{k}$ then $\mathsf{Val}[v_a] \leftarrow b_{\mathsf{pth}}$; $T \leftarrow T \cup \{v_a\}$ ▷ accept leaf

  Else $\mathsf{Val}[v_a] \leftarrow \overline{\mathsf{WC}}[i+1]$

  $a_{prev} \leftarrow x[1..\overline{\mathsf{WC}}[i-1]]$ ▷ previous interior node (If $i = 1$ then $a_{prev} \leftarrow \epsilon$)

  $E \leftarrow E \cup \{(v_{a_{prev}}, v_a, x[j])\}$ ▷ edge labelled with value of current non-* bit

  $a' \leftarrow x[1..(j-1)] \,\|\, (1 - x[j])$ ▷ a' is equal to a with the last bit flipped

  $V \leftarrow V \cup \{v_{a'}\}$; $T \leftarrow T \cup \{v_{a'}\}$; $\mathsf{Val}[v_{a'}] \leftarrow 1 - b_{\mathsf{pth}}$ ▷ reject leaf

  $E \leftarrow E \cup \{(v_{a_{prev}}, v_{a'}, 1 - x[j])\}$ ▷ edge with flipped value of current non-* bit

Return (V, E, T, Val)

Figure 5.2: **Procedure for constructing a branching program from a singleton set containing a λ-bit string with wildcards. See Construction 2 and Section 5.3.**

Overview of SetBP^*. SetBP^* (Fig. 5.2) starts by forming an ordered ascending list of all indices of x without wildcards. Then we loop over each of these indices. A node is added to the BP for every prefix of x ending with an explicit (as opposed to *) bit value. Each node value is set to the index of the next non-wildcard bit in x. The node representing the final non-wildcard index is given value b_{pth}. For example, if $x = 0 * 1 * 0$, then we add prefix nodes $v_\epsilon, v_0, v_{0*1}, v_{0*1*0}$, (where v_ϵ is the root), and set their values to $1, 3, 5, b_{\mathsf{pth}}$, respectively.

Each iteration adds an edge from the previous prefix node to the one just created. This edge is labelled with the bit value at the current non-wildcard index. Continuing with the example, in the iteration node v_{0*1} is created, an edge from v_0 to v_{0*1} is added with label 1. Since S_{R} only contains one element, we also create a reject leaf representing the prefix of the current interior node with the final bit flipped. An edge labelled with this flipped bit is also added from the previous node. In the example, v_{0*0} is created with value $1 - b_{\mathsf{pth}}$ and edge $(v_{0*}, v_{0*0}, 0)$ is added. Once all non-wildcard indices of x have been considered, the BP is returned.

Chapter 6

Conclusion

In this thesis, we further the study of laconic cryptography with a laconic two-party protocol for the secure computation of a possibly unbalanced branching program. Our two-party protocol uses an anonymous garbled circuit scheme with an anonymous hash encryption scheme. Using these tools together allows for the evaluation of the branching program while hiding the intermediate results of the computation.

The protocol construction can be based on either the computational Diffie-Hellman or Learning with Errors assumption. Prior to this work, laconic branching programs could only be realised from the LWE assumption (due to the construction for general functionalities of [QWW18]). Due to the versatility of branching programs, our protocol can be used for laconic PSI and PSU.

Future work. The receiver party's computational cost depends on the number of root-to-leaf paths in the BP. As a result, we are limited to BPs with a polynomial number of such paths. So a direction for future study is to describe which BPs of polynomial size also have a polynomial number of root-to-leaf paths. Within this task is the need to determine if a BP, BP, is *minimal*. Meaning, does there exist BP' with the same functionality as BP while having fewer nodes or paths than BP? For example, all three BPs in Figure 6.1 have different sizes but the same functionality. In this case, BP c) is minimal, but in more complicated BPs, it is not always so clear. These topics were beyond the scope of this thesis but could allow our protocol to have applications in fuzzy matching.

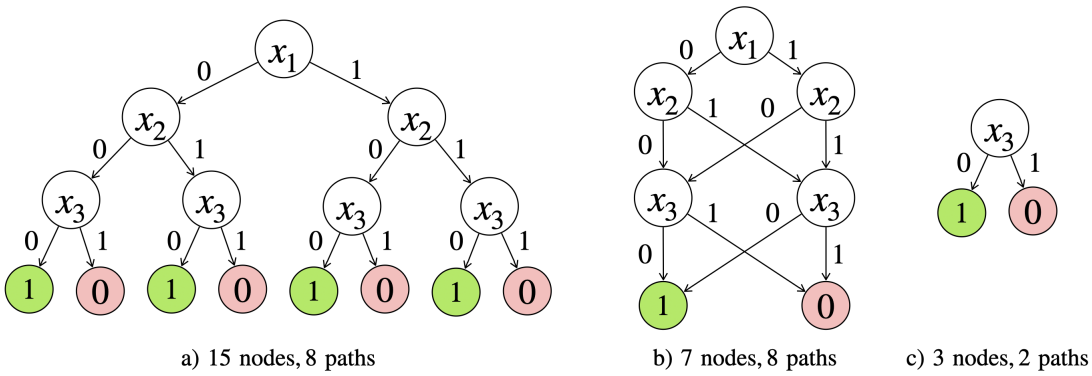


Figure 6.1: **Three branching programs with the same functionality.**

References

[ABD+21] Navid Alamati, Pedro Branco, Nico Döttling, Sanjam Garg, Mohammad Hajiabadi, and Sihang Pu. Laconic private set intersection and applications. In Kobbi Nissim and Brent Waters, editors, *TCC 2021: 19th Theory of Cryptography Conference, Part III*, volume 13044 of *Lecture Notes in Computer Science*, pages 94–125, Raleigh, NC, USA, November 8–11, 2021. Springer, Heidelberg, Germany. iii, viii, 2, 3, 4, 7, 10, 12, 13, 16, 21, 23, 24

[ALOS22] Diego F. Aranha, Chuanwei Lin, Claudio Orlandi, and Mark Simkin. Laconic private set-intersection from pairings. In Heng Yin, Angelos Stavrou, Cas Cremers, and Elaine Shi, editors, *ACM CCS 2022: 29th Conference on Computer and Communications Security*, pages 111–124, Los Angeles, CA, USA, November 7–11, 2022. ACM Press. 2, 3, 4, 7

[BFK+09] Mauro Barni, Pierluigi Failla, Vladimir Kolesnikov, Riccardo Lazzeretti, Ahmad-Reza Sadeghi, and Thomas Schneider. Secure evaluation of private linear branching programs with medical applications. In Michael Backes and Peng Ning, editors, *Computer Security – ESORICS 2009*, pages 424–439, Berlin, Heidelberg, 2009. Springer Berlin Heidelberg. 3

[BLSV18] Zvika Brakerski, Alex Lombardi, Gil Segev, and Vinod Vaikuntanathan. Anonymous IBE, leakage resilience and circular security from new assumptions. In Jesper Buus Nielsen and Vincent Rijmen, editors, *Advances in Cryptology – EUROCRYPT 2018, Part I*, volume 10820 of *Lecture Notes in Computer Science*, pages 535–564, Tel Aviv, Israel, April 29 – May 3, 2018. Springer, Heidelberg, Germany. 6, 8, 9, 10, 16, 17, 18

[BPSW07] Justin Brickell, Donald E. Porter, Vitaly Shmatikov, and Emmett Witchel. Privacy-preserving remote diagnostics. In Peng Ning, Sabrina De Capitani

di Vimercati, and Paul F. Syverson, editors, *ACM CCS 2007: 14th Conference on Computer and Communications Security*, pages 498–507, Alexandria, Virginia, USA, October 28–31, 2007. ACM Press. 3

[CDG+17] Chongwon Cho, Nico Döttling, Sanjam Garg, Divya Gupta, Peihan Miao, and Antigoni Polychroniadou. Laconic oblivious transfer and its applications. In Jonathan Katz and Hovav Shacham, editors, *Advances in Cryptology – CRYPTO 2017, Part II*, volume 10402 of *Lecture Notes in Computer Science*, pages 33–65, Santa Barbara, CA, USA, August 20–24, 2017. Springer, Heidelberg, Germany. 2, 3, 4, 6, 21

[CDPP22] Kelong Cong, Debajyoti Das, Jeongeun Park, and Hilder V. L. Pereira. SortingHat: Efficient private decision tree evaluation via homomorphic encryption and transciphering. In Heng Yin, Angelos Stavrou, Cas Cremers, and Elaine Shi, editors, *ACM CCS 2022: 29th Conference on Computer and Communications Security*, pages 563–577, Los Angeles, CA, USA, November 7–11, 2022. ACM Press. 3

[CGH+21] Melissa Chase, Sanjam Garg, Mohammad Hajiabadi, Jialin Li, and Peihan Miao. Amortizing rate-1 OT and applications to PIR and PSI. In Kobbi Nissim and Brent Waters, editors, *TCC 2021: 19th Theory of Cryptography Conference, Part III*, volume 13044 of *Lecture Notes in Computer Science*, pages 126–156, Raleigh, NC, USA, November 8–11, 2021. Springer, Heidelberg, Germany. viii, 3, 43

[DG17] Nico Döttling and Sanjam Garg. Identity-based encryption from the Diffie-Hellman assumption. In Jonathan Katz and Hovav Shacham, editors, *Advances in Cryptology – CRYPTO 2017, Part I*, volume 10401 of *Lecture Notes in Computer Science*, pages 537–569, Santa Barbara, CA, USA, August 20–24, 2017. Springer, Heidelberg, Germany. 4, 6, 8, 9, 10, 16

[DGGM19] Nico Döttling, Sanjam Garg, Vipul Goyal, and Giulio Malavolta. Laconic conditional disclosure of secrets and applications. In David Zuckerman, editor, *60th Annual Symposium on Foundations of Computer Science*, pages 661–685, Baltimore, MD, USA, November 9–12, 2019. IEEE Computer Society Press. 3, 12

[DGI+19] Nico Döttling, Sanjam Garg, Yuval Ishai, Giulio Malavolta, Tamer Mour, and Rafail Ostrovsky. Trapdoor hash functions and their applications. In

Alexandra Boldyreva and Daniele Micciancio, editors, *Advances in Cryptology – CRYPTO 2019, Part III*, volume 11694 of *Lecture Notes in Computer Science*, pages 3–32, Santa Barbara, CA, USA, August 18–22, 2019. Springer, Heidelberg, Germany. 3, 6

[DKL+23] Nico Döttling, Dimitris Kolonelos, Russell W. F. Lai, Chuanwei Lin, Giulio Malavolta, and Ahmadreza Rahimi. Efficient laconic cryptography from learning with errors. In Carmit Hazay and Martijn Stam, editors, *Advances in Cryptology – EUROCRYPT 2023, Part III*, volume 14006 of *Lecture Notes in Computer Science*, pages 417–446, Lyon, France, April 23–27, 2023. Springer, Heidelberg, Germany. 7

[Gen09] Craig Gentry. Fully homomorphic encryption using ideal lattices. In Michael Mitzenmacher, editor, *41st Annual ACM Symposium on Theory of Computing*, pages 169–178, Bethesda, MD, USA, May 31 – June 2, 2009. ACM Press. 3

[GGH19] Sanjam Garg, Romain Gay, and Mohammad Hajiabadi. New techniques for efficient trapdoor functions and applications. In Yuval Ishai and Vincent Rijmen, editors, *Advances in Cryptology – EUROCRYPT 2019, Part III*, volume 11478 of *Lecture Notes in Computer Science*, pages 33–63, Darmstadt, Germany, May 19–23, 2019. Springer, Heidelberg, Germany. 17

[GHMM24a] Sanjam Garg, Mohammad Hajiabadi, Peihan Miao, and Alice Murphy. Laconic branching programs from the diffie-hellman assumption. Cryptology ePrint Archive, Paper 2024/102, 2024. https://eprint.iacr.org/2024/102. iii

[GHMM24b] Sanjam Garg, Mohammad Hajiabadi, Peihan Miao, and Alice Murphy. Laconic branching programs from the diffie-hellman assumption. In Qiang Tang and Vanessa Teague, editors, *Public-Key Cryptography – PKC 2024*, pages 323–355, Cham, 2024. Springer Nature Switzerland. iii

[GHMR18] Sanjam Garg, Mohammad Hajiabadi, Mohammad Mahmoody, and Ahmadreza Rahimi. Registration-based encryption: Removing private-key generator from IBE. In Amos Beimel and Stefan Dziembowski, editors, *TCC 2018: 16th Theory of Cryptography Conference, Part I*, volume 11239 of *Lecture Notes in Computer Science*, pages 689–718, Panaji, India, November 11–14, 2018. Springer, Heidelberg, Germany. 10

[GHO20] Sanjam Garg, Mohammad Hajiabadi, and Rafail Ostrovsky. Efficient range-trapdoor functions and applications: Rate-1 OT and more. In Rafael Pass and Krzysztof Pietrzak, editors, *TCC 2020: 18th Theory of Cryptography Conference, Part I*, volume 12550 of *Lecture Notes in Computer Science*, pages 88–116, Durham, NC, USA, November 16–19, 2020. Springer, Heidelberg, Germany. 3

[GMW87] Oded Goldreich, Silvio Micali, and Avi Wigderson. How to play any mental game or A completeness theorem for protocols with honest majority. In Alfred Aho, editor, *19th Annual ACM Symposium on Theory of Computing*, pages 218–229, New York City, NY, USA, May 25–27, 1987. ACM Press. 1

[GRS22] Gayathri Garimella, Mike Rosulek, and Jaspal Singh. Structure-aware private set intersection, with applications to fuzzy matching. In Yevgeniy Dodis and Thomas Shrimpton, editors, *Advances in Cryptology – CRYPTO 2022, Part I*, volume 13507 of *Lecture Notes in Computer Science*, pages 323–352, Santa Barbara, CA, USA, August 15–18, 2022. Springer, Heidelberg, Germany. 7

[GSW13] Craig Gentry, Amit Sahai, and Brent Waters. Homomorphic encryption from learning with errors: Conceptually-simpler, asymptotically-faster, attribute-based. In Ran Canetti and Juan A. Garay, editors, *Advances in Cryptology – CRYPTO 2013, Part I*, volume 8042 of *Lecture Notes in Computer Science*, pages 75–92, Santa Barbara, CA, USA, August 18–22, 2013. Springer, Heidelberg, Germany. 2, 3

[IP07] Yuval Ishai and Anat Paskin. Evaluating branching programs on encrypted data. In Salil P. Vadhan, editor, *TCC 2007: 4th Theory of Cryptography Conference*, volume 4392 of *Lecture Notes in Computer Science*, pages 575–594, Amsterdam, The Netherlands, February 21–24, 2007. Springer, Heidelberg, Germany. 2, 3, 15

[KNL+19] Ágnes Kiss, Masoud Naderpour, Jian Liu, N. Asokan, and Thomas Schneider. SoK: Modular and efficient private decision tree evaluation. *Proceedings on Privacy Enhancing Technologies*, 2019(2):187–208, April 2019. 3

[QWW18] Willy Quach, Hoeteck Wee, and Daniel Wichs. Laconic function evaluation and applications. In Mikkel Thorup, editor, *59th Annual Symposium on Foundations of Computer Science*, pages 859–870, Paris, France, October 7–9, 2018. IEEE Computer Society Press. 2, 3, 6, 47

[Rab81] Michael O. Rabin. How to exchange secrets with oblivious transfer. Technical report, TR-81, Aiken Computation Lab, Harvard University, 1981. 2

[Rab05] Michael O. Rabin. How to exchange secrets with oblivious transfer. Cryptology ePrint Archive, Report 2005/187, 2005. https://eprint.iacr.org/2005/187. 2

[Yao82] Andrew C. Yao. Protocols for secure computations. In *2013 IEEE 54th Annual Symposium on Foundations of Computer Science*, pages 160–164, Los Alamitos, CA, USA, nov 1982. IEEE Computer Society. 1

[Yao86] Andrew Chi-Chih Yao. How to generate and exchange secrets (extended abstract). In *27th Annual Symposium on Foundations of Computer Science*, pages 162–167, Toronto, Ontario, Canada, October 27–29, 1986. IEEE Computer Society Press. 1, 2, 8

APPENDICES

Appendix A

Supplementary Figures

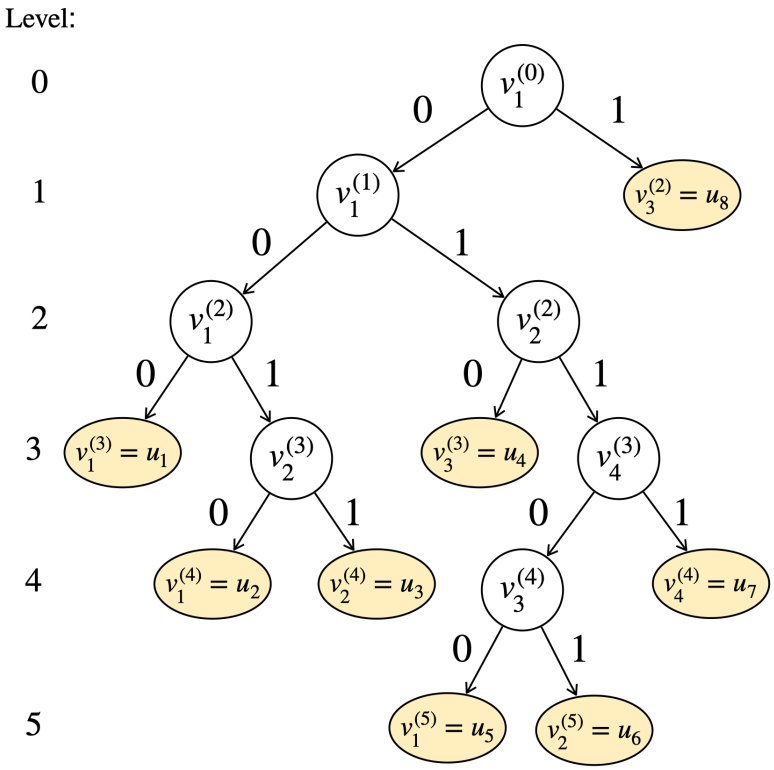


Figure A.1: **Example of the node labelling conventions used throughout the paper.**

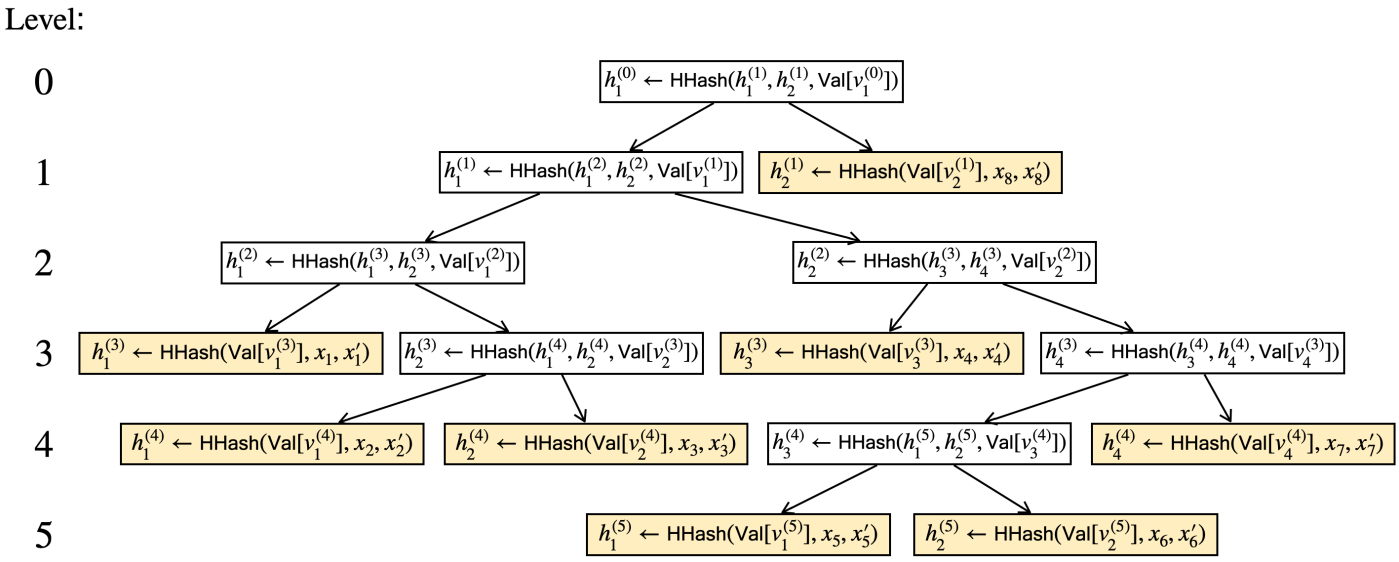


Figure A.2: **The hashing procedure notation demonstrated on the Figure A.1 BP.**

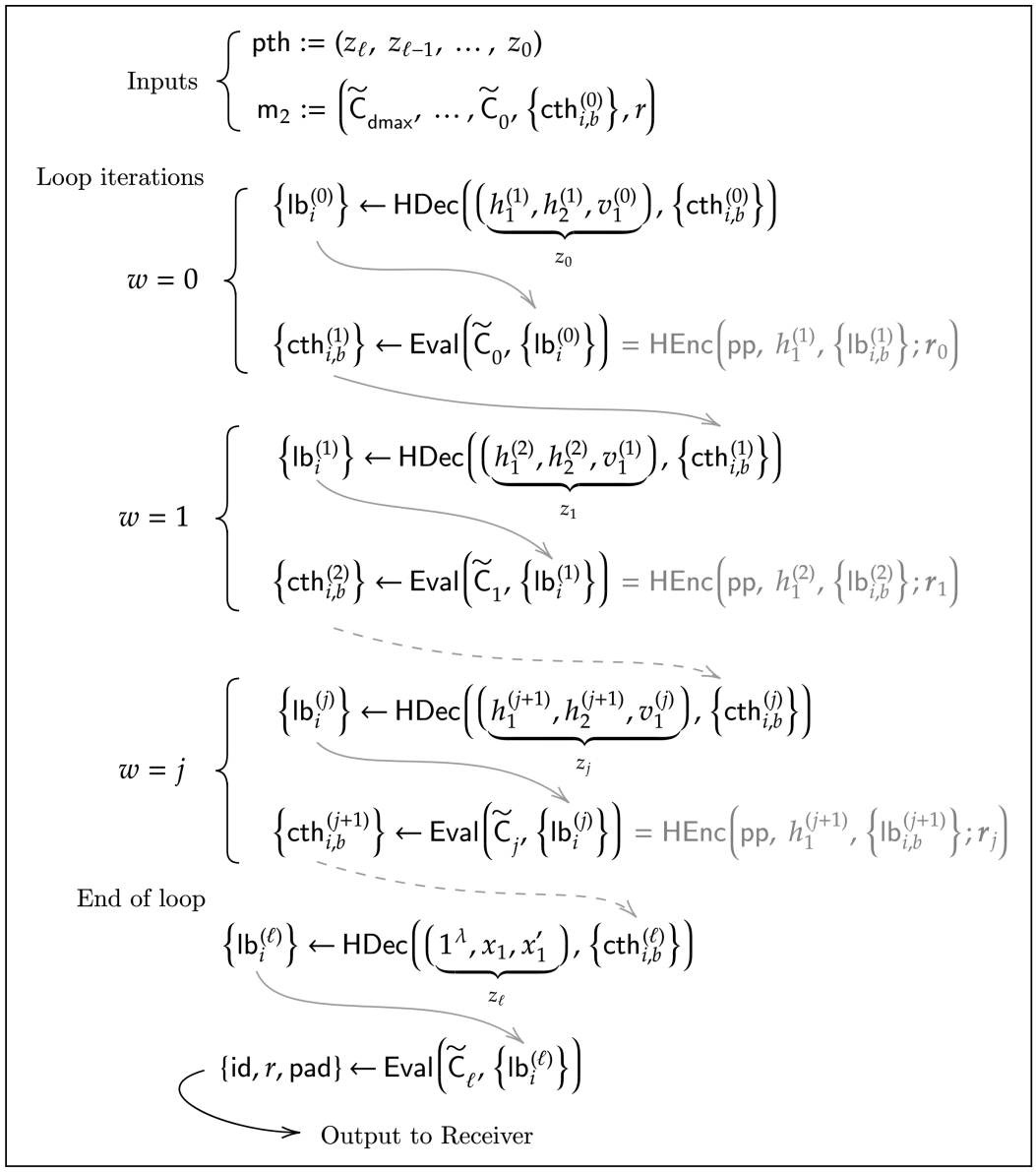


Figure A.3: **Illustration of the progression of** DecPath **given in Figure 4.2, assuming the input path has endpoint the leftmost leaf and has value** 1^λ**.**